Structural Analysis-I
(23HPC0108)

LECTURE NOTES

II-B.TECH & II-SEM

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II Year B.Tech. CE – II Semester

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(23HPC0108) STRUCTURAL ANLAYSIS

Course Objectives

Learn energy theorems
Learn the analysis of indeterminate structures
Analysis of fixed and continuous beams
Learn about slope-deflection method
Learn about Moment – distribution method

Course Outcomes:

- Apply energy theorems to analyze trusses
- Analyze indeterminate structures by using Castigliano's–II theorem
- Analysis of fixed and continuous beams
- Analyze continuous beams and portal frames by using slope-deflection method
- Analyze continuous beams and portal frames by using Moment – distribution method

UNIT – I

ENERGY THEOREMS: Introduction-Strain energy in linear elastic system, expression of strain energy due to axial load, bending moment and shear forces – Castigliano's first theoremDeflections of simple beams and pin jointed trusses.

UNIT - II

ANALYSIS OF INDETERMINATE STRUCTURES: Indeterminate Structural Analysis – Determination of static and kinematic indeterminacies – Solution of trusses with upto two degrees of internal and external indeterminacies – Castigliano's–II theorem.

UNIT - III

FIXED BEAMS & CONTINUOUS BEAMS : Introduction to statically indeterminate beams with uniformly distributed load, central point load, eccentric point load, number of point loads, uniformly varying load, couple and combination of loads – Shear force and Bending moment diagrams – Deflection of fixed beams effect of sinking of support, effect of rotation of a support.

UNIT - IV

SLOPE-DEFLECTION METHOD: Introduction-derivation of slope deflection equations-application to continuous beams with and without settlement of supports - Analysis of single bayportal frames without sway.

JNTU Hyd. Raghukanth

UNIT - V

MOMENT DISTRIBUTION METHOD: Introduction to moment distribution method-Application to continuous beams with and without settlement of supports-Analysis of single bay storey portal frames without sway.

Textbooks:

1. Analysis of Structures – Vol-I&II by V.N.Vazirani&M.M.Ratwani, Khanna Publications, New Delhi.
2. Basic Structural Analysis by C.S.Reddy., Tata McGraw Hill Publishers. 3rd edition 2017.

Reference Books:

1. Structural analysis by Aslam Kassimali Cengage publications 6th edition 2020.
2. Structural analysis Vol.I and II by Dr.R.Vaidyanathan and Dr.PPerumal– Laxmi publications. 3rd 2016
3. Introduction to structural analysis by B.D.Nautiyal, New Age international publishers, New Delhi.
4. Structural Analysis – D.S.Prakasarao -Univeristy press.
5. Strength of Materials and Mechanics of Structures by B.C.Punmia, Khanna Publications, New Delhi.

UNIT-①

ENERGY METHODS

Introduction:-

When an external load act on a structure, the structure undergoes deformation & hence, the work is done. To resist these external forces, the internal forces develop gradually from zero to their final value & work is done. This internal workdone is stored as energy in the structure & it helps the structure to spring back to the original shape & size, whenever the external loads are removed, provided the material of the structure is still within elastic limit.

When equilibrium is reached, as per the well known law of Conservation of energy, the work done by the external forces must equal the strain energy stored. This concept of energy balance is utilized in structural analysis to develop a number of methods to find deflection of structures. The following methods are finding the deflection of beams & frames.

① Strain Energy / Real work method

② Virtual work / Unit load Method

③ Castigliand's Method.

Strain Energy:-

When an elastic body is subjected to external forces it will deform, if the elastic limit is not ~~reachable~~ exceeded, the work done in straining the material is stored in the form of resilience. of internal energy. This is known as "strain Energy".

(or)

Internal energy stored in a body within elastic limit of elastic body is known as Strain Energy.

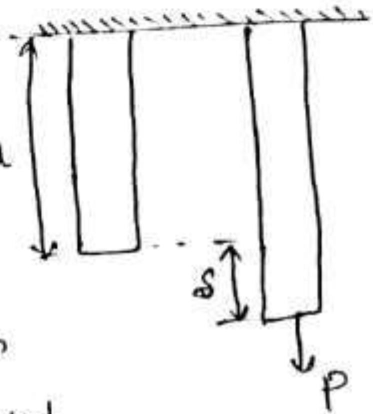
Strain Energy due to axial loading:-

Let l = length of member,

A = Area of c/s of member

δ = Extension of member

P = External load



Since the load is applied gradually from zero to 'P' the member is also gradually extended.

$\therefore$ External workdone by force (W_e) = Average force × distance

$$= \left(\frac{0+P}{2}\right) \times \delta$$

$$W_e = \frac{1}{2} \times P \times \delta \longrightarrow ①$$

Let Internal workdone (or) Strain Energy = $W_i = U \longrightarrow ②$

From law of Conservation of Energy

Internal workdone = External work done

i.e. eq① = eq②

$$U = \frac{1}{2} \times P \times \delta \longrightarrow ③$$

But we know $\delta = \frac{Pl}{AE} \longrightarrow ④$ $\left(\because \sigma \propto e \quad \frac{P}{A} = E \cdot \frac{\delta}{l} \quad \delta = \frac{Pl}{AE}\right)$

From eq③ & eq④ $U = \frac{1}{2} \times P \times \frac{Pl}{AE}$

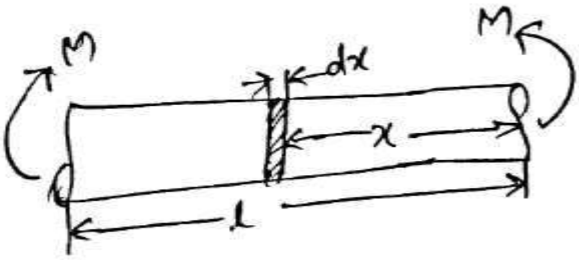
$$\boxed{U = \frac{P^2 l}{2AE}}$$

If, however, the bar has variable area of c/s, consider a small section of length dx & area of c/s A. The strain Energy in small element of length dx, is $dU = \frac{P^2 dx}{2AE}$

Total strain Energy $\boxed{U = \int_0^l \frac{P^2 dx}{2AE}}$

Strain Energy due to Bending Moment:-

Consider a member of length(l) Subjected to uniform bending moment(M). In that Consider a small element of length "dx". Let '$d\theta$' is the change in slope.



So strain Energy stored in the element

$$du = \frac{1}{2} \times M \times d\theta \longrightarrow (1)$$

But we know $\frac{M}{EI} = \frac{d^2y}{dx^2}$

$$\frac{M}{EI} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$\frac{M}{EI} = \frac{d}{dx}(\theta) = \frac{d\theta}{dx} \qquad \left(\because \theta = \frac{dy}{dx}\right)$$

$$\Rightarrow d\theta = \frac{M}{EI} \times dx \longrightarrow (2)$$

From eq (1) & eq (2)

$$du = \frac{1}{2} \times M \times \frac{M}{EI} dx$$

$$\boxed{du = \frac{M^2 dx}{2EI}}$$

The total strain Energy

$$U = \int_0^l \frac{M^2 dx}{2EI}$$

(or)

$$U = \frac{1}{2EI} \cdot \int_0^l M^2 dx$$

Strain Energy due to Torsion.

Consider a shaft of length (l) subjected to twisting moment (T). When torsion is subjected to shaft it will produce twist. Let 'θ' be the angle of twist.

∴ Work done by External force (W_E) $= \frac{1}{2} \times T \times \theta$ ⟶ (1)

Internal work done (or) Strain Energy (W_I) = U ⟶ (2)

From law of Conservation of energy

eq(1) = eq(2)

$$U = \frac{1}{2} \times T \times \theta \longrightarrow (3)$$

But we know from torsion Equation

$$\frac{T}{J} = \frac{G\theta}{l} = \frac{\tau}{R}$$

$$\frac{T}{J} = \frac{G\theta}{l}$$

$$\theta = \frac{Tl}{GJ} \longrightarrow (4)$$

From eq(3) & eq(4)

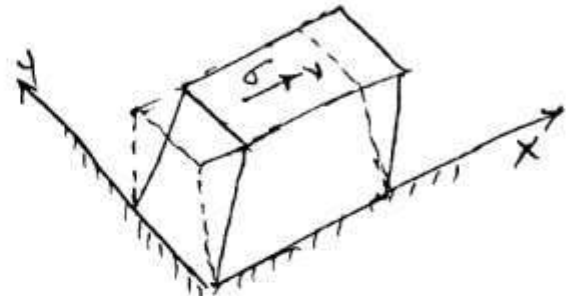
$$U = \frac{1}{2} \times T \times \frac{Tl}{GJ}$$

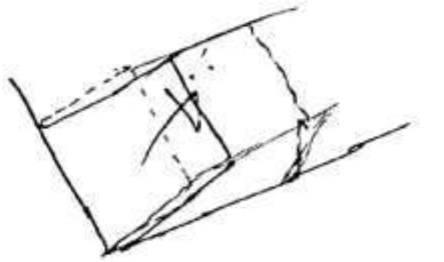
$$\boxed{U = \frac{1}{2} \times \frac{T^2 l}{GJ}}$$

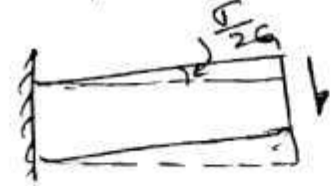
(or)

$$\boxed{U = \int_0^l \frac{T^2\,dx}{2GJ}}$$

Strain Energy due to transverse Shear:-







The shear stress on a c/s of beam of rectangular c/s may be found out by the relation

$$\tau = \frac{VQ}{bI_{zz}}$$

where Q = first moment of portion of c/s above the point where shear stress is reqd about NA

V = Transverse shear force

b = width of section.

I_{zz} = MOI of the section about NA.

due to shearstress, the angle b/w the lines of right angle will change

The shear stress varies across the height in a parabolic manner in the case of rectangular c/s. Also, the shear stress distribution is different for different shape of c/s. However, to simplify the computation of shear stress is assumed to be uniform across the c/s. Consider segment of length "dx" subjected to shear stress τ. The shear stress across the c/s may be taken as

$$\tau = k \times \frac{V}{A} \rightarrow (1)$$ where k = factor, depend on shape of c/s

deformation $d\delta = \Delta\gamma \cdot dx \rightarrow (2)$

But we know $G = \frac{\tau}{\Delta\gamma} = \frac{\text{shear stress}}{\text{shear strain}}$

$$\Rightarrow \Delta\gamma = \frac{\tau}{G} \rightarrow (3)$$

from eq ① & ②

$$\Delta\gamma = k \times \frac{V}{AG}$$

From eq ②, deformation $d\delta = k \cdot \frac{V}{AG} \times dx$

Total deformation $\delta = \int_0^l k \cdot \frac{V}{AG} \cdot dx.$

Strain Energy $U = \frac{1}{2} \times V \times \delta$

$$U = \frac{1}{2} \times V \times \int_0^l k \cdot \frac{V^2}{AG} dx$$

$$\boxed{U = \int_0^l \frac{kV^2}{2AG} \cdot dx}$$

Strain Energy :-

① Due to axial loading $= \int_0^l \frac{P^2}{2AE} \cdot dx$

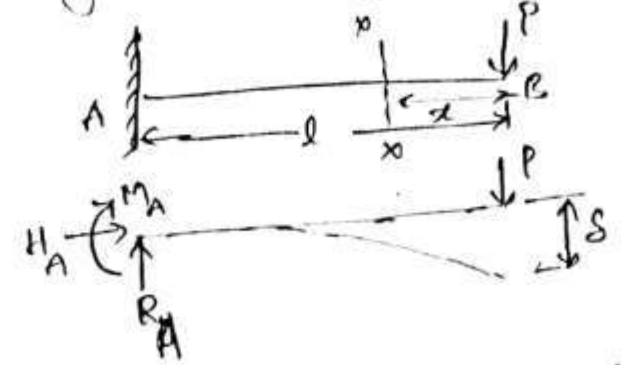
② Due to Bending $= \int_0^l \frac{M^2}{2EI} \cdot dx$

③ Due to twisting $= \int_0^l \frac{T^2}{2GJ} \cdot dx$

④ Due to shear $= \int_0^l \frac{V^2}{2AG} \cdot dx$

Problem: find the deflection at free end of Cantilever carrying a point load at free end using strain energy principle.

Sol:-



Now BM at section x-x from free end $M = Px$

strain Energy $U = \int_0^l \frac{M^2 dx}{2EI}$

$= \int_0^l \frac{(Px)^2 dx}{2EI} = \int_0^l \frac{P^2 x^2 dx}{2EI}$

$= \frac{P^2}{2EI} \cdot \int_0^l x^2 \cdot dx$

$= \frac{P^2}{2EI} \left(\frac{x^3}{3}\right)_0^l = \frac{P^2}{2EI}\left(\frac{l^3}{3} - 0\right)$

$U = \frac{P^2 l^3}{6EI} \longrightarrow ②$

workdone by External load $= \frac{1}{2} \times P \times \delta \longrightarrow ②$

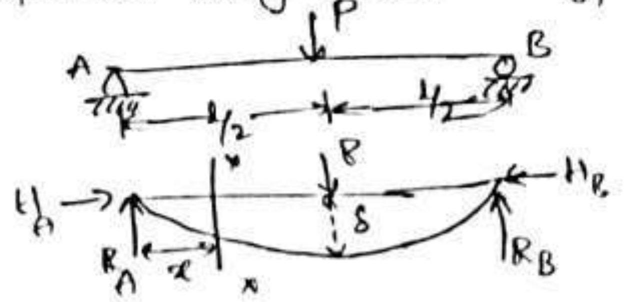
eq① = eq②

$\frac{P^2 l^3}{6EI} = \frac{1}{2} \times P \times \delta$

$$\boxed{\delta = \frac{Pl^3}{3EI}}$$

Prob②:- A Beam of span 'l'. Carries a Concentrated load 'P'. at midspan find Central deflection using strain energy principle.

Sol:-



Reaction at each end $R_A = R_B = \frac{P}{2}$

Bending Moment at section x-x, $M = R_A \times x = \frac{P}{2} \times x$

Strain Energy stored by half of the beam $= \int_0^{l/2} \frac{M^2}{2EI} dx$

Total Strain Energy $U = 2\int_0^{l/2} \frac{M^2}{2EI} . dx$

$$= 2 \int_0^{l/2} \left(\frac{P}{2}x\right)^2 \times \frac{1}{2EI} dx$$

$$= \frac{2}{2EI} \int_0^{l/2} \frac{P^2}{4} \times x^2 . dx$$

$$= \frac{P^2}{4EI} \int_0^{l/2} x^2 . dx = \frac{P^2}{4EI}\left(\frac{x^3}{3}\right)_0^{l/2}$$

$$= \frac{P^2}{4EI}\left[\frac{(l/2)^3}{3} - 0\right]$$

$$= \frac{P^2}{4EI}\left[\frac{l^3}{8 \times 3}\right]$$

$$U = \frac{P^2 l^3}{96EI} \longrightarrow (1)$$

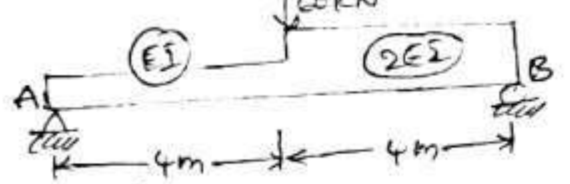
work done by External force $= \frac{1}{2} \times P \times \delta \longrightarrow (2)$

eq (1) = eq (2)

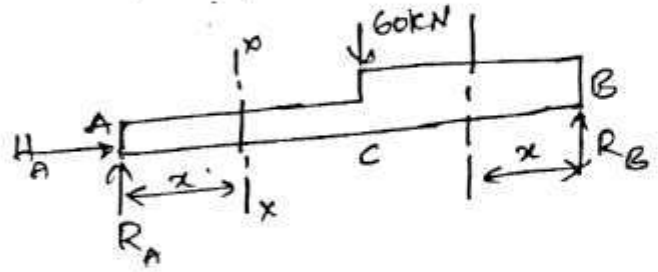
$$\frac{P^2 l^3}{96EI} = \frac{1}{2} \times P \times \delta$$

$$\boxed{\delta = \frac{Pl^3}{48EI}}$$

Problem ⑤ :- Determine deflection under 60KN load by using Strain Energy Method.



Sol:-



Reactions at each support $R_A = R_B = \frac{60}{2} = 30KN$

BM at Section x–x from Support A = $R_A x = 30x$

BM at Section x–x from Support B = $R_B x = 30x$

Strain Energy Stored in portion AC = $\int_0^4 \frac{M^2}{2EI} dx = \int_0^4 \frac{(30x)^2}{2EI} dx$

Strain Energy Stored in portion BC = $\int_0^4 \frac{M^2}{2(2EI)} dx = \int_0^4 \frac{(30x)^2}{4EI} dx$

Total Strain Energy stored in member

$$U = \int_0^4 \frac{(30x)^2}{2EI} dx + \int_0^4 \frac{(30x)^2}{4EI} dx$$

$$= \frac{\overset{450}{900}}{2EI} \int_0^4 x^2 dx + \frac{900}{4EI} \int_0^4 x^2 dx$$

$$= \frac{450}{EI} \left[\frac{x^3}{3}\right]_0^4 + \frac{225}{EI} \left(\frac{x^3}{3}\right)_0^4$$

$$= \frac{450}{EI}\left(\frac{64}{3}\right) + \frac{225}{EI}\left(\frac{64}{3}\right)$$

$$U = \frac{14400}{EI} \longrightarrow ①$$

But External work done $W_e = \frac{1}{2} \times P \times \delta = \frac{1}{2} \times \overset{30}{60} \times \delta = 30\delta \longrightarrow ②$

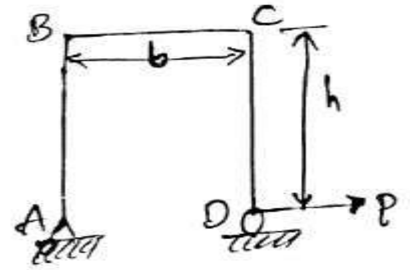
eq① = eq③

$$\frac{14400}{EI} = 30\delta$$

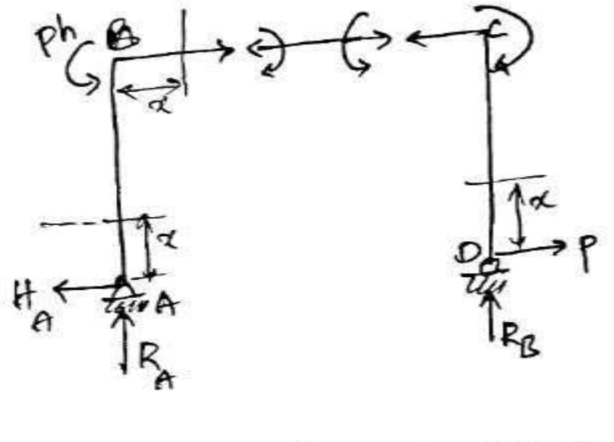
$$\delta = \frac{14400}{30EI}$$

$$\boxed{\delta = \frac{480}{EI}}$$

Prob(?) :- A portal frame ABCD has it's end 'A' is hinged while end 'D' is ~~pinned~~ placed on roller a horizontal force 'P' is applied on the end 'D' as shown in fig. Determine horizontal movement of 'D'. Assume the members have same flexural rigidity.



Sol:-



$R_A = R_B = 0,$

$H_A = P$

From above fig. the BM expression for various portion are

portion	AB	BC	CD
origin	A	B	D
Limit	0-h	0-b	0-h
M_x	$H_A x = Px$	Ph	Px

Total strain Energy stored in frame $U = \int_0^h \frac{M^2}{2EI}dx + \int_0^b \frac{M^2}{2EI}dx + \int_0^h \frac{M^2}{2EI}dx$

$$U = \int_0^h \frac{(Px)^2}{2EI}dx + \int_0^b \frac{(Ph)^2}{2EI}dx + \int_0^h \frac{(Px)^2}{2EI}dx.$$

$$= 2\int_0^h \frac{P^2x^2}{2EI}dx + \int_0^b \frac{P^2h^2}{2EI}dx$$

$$= \frac{2P^2}{2EI}\int_0^h x^2 dx + \frac{P^2h^2}{2EI}\int_0^b 1\,dx.$$

$$= \frac{P^2}{EI}\left(\frac{x^3}{3}\right)_0^h + \frac{P^2h^2}{2EI}\left(x\right)_0^b$$

$$= \frac{P^2}{3EI}(h^3) + \frac{P^2h^2}{2EI}(b)$$

$$= \frac{P^2h^2}{EI}\left(\frac{h}{3}+\frac{b}{2}\right) = \frac{P^2h^2}{EI}\left(\frac{2h+3b}{6}\right)$$

$$U = \frac{P^2h^2}{6EI}(2h+3b) \longrightarrow ①$$

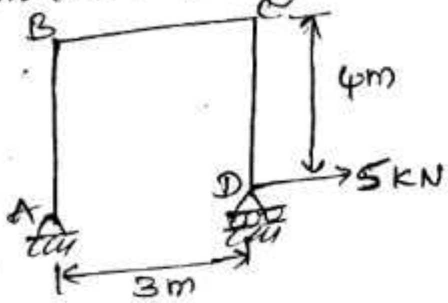
External workdone $(W_e) = \frac{1}{2} \times P \times \delta \longrightarrow ②$

$$eq① = eq②$$

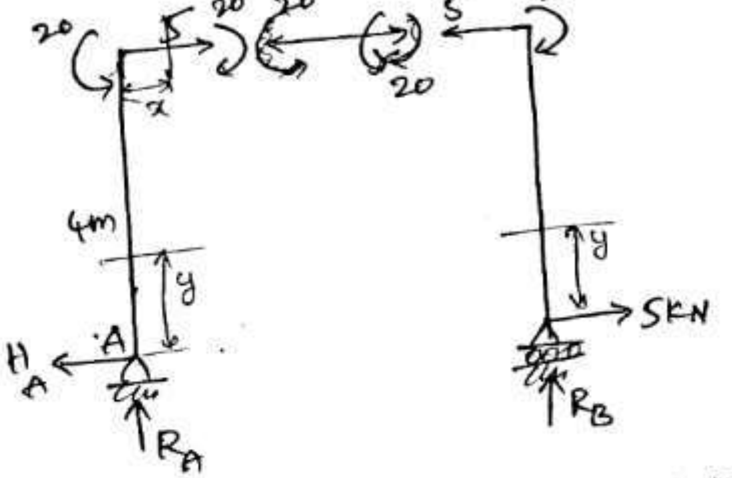
$$\frac{1}{2} \times P \times \delta = \frac{P^2 h^2}{6EI}(2h+3b)$$

$$\boxed{\delta = \frac{Ph^2}{3EI}(2h+3b)}$$

prob :- Determine the horizontal displacement of the roller end 'D' of the portal frame shown in fig. $EI = 8000\ kN\text{-}m^2$ throughout.



Sol:-



$R_A = R_B = 0.$

$H_A = 5\,kN.$

from the above the BM expression for various portions are

portion	AB	BC	CD
origin	A	B	D
Limit	0–4	0–3	0–4
M_x	$5x$	20	$5x$

Total Strain Energy stored in frame $U = \int_0^4 \frac{M^2}{2EI}dx + \int_0^3 \frac{M^2}{2EI}dx + \int_0^4 \frac{M^2}{2EI}dx$

$$U = \int_0^4 \frac{(5x)^2}{2EI}dx + \int_0^3 \frac{(20)^2}{2EI}dx + \int_0^4 \frac{(5x)^2}{2EI}dx$$

$$= 2\int_0^4 \frac{(5x)^2 dx}{2EI} + \int_0^3 \frac{400}{2EI}dx$$

$$= \frac{2 \times 25}{2EI}\int_0^4 x^2 dx + \frac{400}{2EI}\int_0^3 1\cdot dx$$

$$= \frac{25}{EI}\left(\frac{x^3}{3}\right)_0^4 + \frac{200}{EI}\left(x\right)_0^3$$

$$U = \frac{25}{EI}\left(\frac{64}{3}\right) + \frac{200}{EI}(3)$$

$$= \frac{25}{EI}\left(\frac{64}{3} + 24\right) = \frac{25}{EI}\left(\frac{64+72}{3}\right)$$

$$= \frac{25}{EI}\left(\frac{136}{3}\right)$$

$$U = \frac{1133.33}{EI} \rightarrow ①$$

External workdone $W_e = \frac{1}{2} \times P \times \delta \rightarrow ②$

eq ① = eq ②

$$\frac{1}{2} \times P \times \delta = \frac{1133.33}{EI}$$

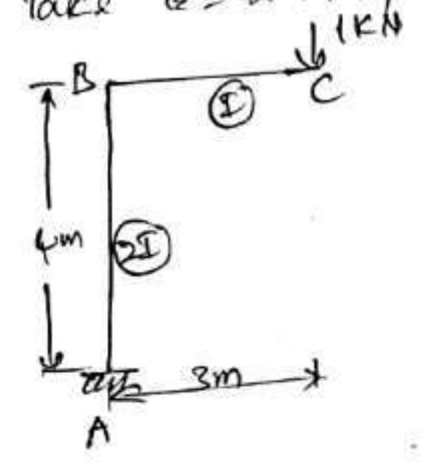
$$\frac{1}{2} \times 5 \times \delta = \frac{1133.33}{EI} \quad (2.5)$$

$$\delta = \frac{1133.33}{2.5EI} = \frac{1133.33}{2.5 \times 8000}$$

$$\delta = 0.05667 \, m$$

$$\boxed{\delta = 56.67 \, mm}$$

Prob 6 Determine the vertical deflection at point 'C'. In the frame shown in fig. Take ~~E = 200 KN/~~ ~~KN-mm²~~ $E = 200 \, KN/mm^2$, $I = 30 \times 10^6 \, mm^4$.



Strain Energy method can be conveniently used for finding deflection in structures only under the following conditions.

1. The structure is subjected to single concentrated load.
2. Deflection reqd is at the ~~end~~ loaded point end only & is in direction of the load.

Castigliano's first theorem:-

Statement:- In a linear elastic structure, partial derivative of the strain energy with respect to a load is equal to the deflection of the point where the load is acting, the deflection being measured in the direction of load.

The load may be force (or) moment. Mathematically this theorem may be represented by

$$\frac{\partial U}{\partial P_i} = \Delta_i \quad ; \quad \frac{\partial U}{\partial M_j} = \theta_j$$

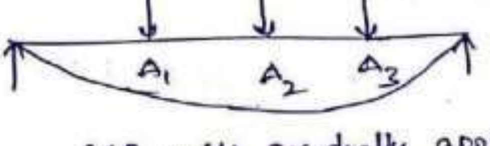
where U = Total Strain Energy

P_i, M_j = load.

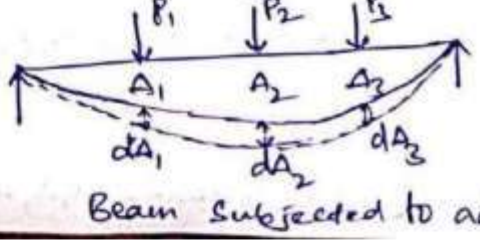
Δ_i, θ_j = deflections.

proof:-

Consider a SSB shown in fig on which loads P_1, P_2 & P_3 are applied gradually. Let deflections under the loads P_1, P_2, P_3 be Δ_1, Δ_2 & Δ_3 respectively.



SSB with gradually applied loads.



Beam subjected to additional load.

Total Strain Energy $U = \frac{1}{2}P_1\Delta_1 + \frac{1}{2}P_2\Delta_2 + \frac{1}{2}P_3\Delta_3 \rightarrow (1)$

Let, the additional load dP_1 be added after the loads P_1, P_2 & P_3

Let the additional deflection be $d\Delta_1, d\Delta_2$ & $d\Delta_3$

Additional Strain Energy $dU = \frac{1}{2}dP_1 d\Delta_1 + P_1 d\Delta_1 + P_2 d\Delta_2 + P_3 d\Delta_3$

$$dU = P_1 d\Delta_1 + P_2 d\Delta_2 + P_3 d\Delta_3 \rightarrow (2)$$

Total Strain Energy

$$U + dU = \frac{1}{2}P_1\Delta_1 + \frac{1}{2}P_2\Delta_2 + \frac{1}{2}P_3\Delta_3 + \frac{1}{2}dP_1 d\Delta_1 + P_1 d\Delta_1 + P_2 d\Delta_2 + P_3 d\Delta_3 \rightarrow (3)$$

If $(P_1 + dP_1)$, P_2 & P_3 were applied simulatiniously strain Energy stored.

$$= \frac{1}{2}(P_1 + dP_1)(\Delta_1 + d\Delta_1) + \frac{1}{2}P_2(\Delta_2 + d\Delta_2) + \frac{1}{2}P_3(\Delta_3 + d\Delta_3) \rightarrow (4)$$

eq (3) = eq (4)

$$U + dU = \left[\frac{1}{2}(P_1 + dP_1)(\Delta_1 + d\Delta_1)\right] + \frac{1}{2}\left[P_2\Delta_2 + d\Delta_2 P_2\right] + \frac{1}{2}\left[P_3\Delta_3 + P_3 d\Delta_3\right]$$

$$= \frac{1}{2}\left[P_1\Delta_1 + P_1 d\Delta_1 + dP_1\Delta_1 + dP_1 d\Delta_1\right] + \frac{1}{2}P_2\Delta_2 + \frac{1}{2}P_2 d\Delta_2 + \frac{1}{2}P_3\Delta_3 + \frac{1}{2}P_3 d\Delta_3$$

$$= \frac{1}{2}P_1\Delta_1 + \frac{1}{2}P_1 d\Delta_1 + \frac{1}{2}dP_1\Delta_1 + \frac{1}{2}dP_1 d\Delta_1 + \frac{1}{2}P_2\Delta_2 + \frac{1}{2}P_2 d\Delta_2 + \frac{1}{2}P_3\Delta_3 + \frac{1}{3}P_3 d\Delta_3$$

$$\cancel{U} + dU = \cancel{U} + \frac{1}{2}P_1 d\Delta_1 + \frac{1}{2}dP_1\Delta_1 + \frac{1}{2}P_2 d\Delta_2 + \frac{1}{2}P_3 d\Delta_3$$

$$dU = \frac{1}{2}\left[P_1 d\Delta_1 + P_2 d\Delta_2 + P_3 d\Delta_3 + dP_1\Delta_1\right]$$

$$2dU = \left[dU + dP_1\Delta_1\right]$$

$$dU = dP_1\Delta_1$$

$$\boxed{\frac{dU}{dP_1} = \Delta_1}$$ simillarlly

$$\frac{dU}{dP_2} = \Delta_2 - - - -$$

Example 3.11 A simply supported beam of span L, carries a concentrated load P at a distance a from left hand side support as shown in Figure 3.22. Using castigliano's theorem determine the deflection under the load. Assume uniform flexural rigidity.

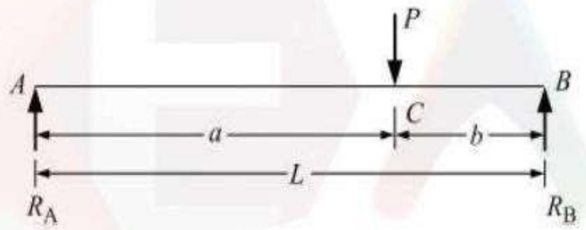


Figure 3.22 Example 3.11

Solution Reaction at A,

$$R_A = \frac{Pb}{L}$$

and Reaction at B,

$$R_B = \frac{Pa}{L}$$

Table 3.9 Calculation table for Example 3.11

Portion	*AC*	*CB*
Origin	A	B
Limit	$0–a$	$0–b$
M	$\frac{Pb}{L}x$	$\frac{Pa}{L}x$
Flexural Rigidity	EI	EI

Therefore, Shear Energy of the beam

$$U = \int_0^a \left(\frac{Pb}{L}x\right)^2 \times \frac{1}{2EI}dx + \int_0^b \left(\frac{Pb}{L}x\right)^2 \times \frac{1}{2EI}dx$$

$$= \left[\frac{P^2b^2}{L^2} \times \frac{1}{6EI}x^3\right]_0^a + \left[\frac{P^2a^2}{L^2} \times \frac{1}{6EI}x^3\right]_0^b$$

$$= \frac{P^2b^2a^3}{6EIL^2} + \frac{P^2a^2b^3}{6EIL^2}$$

$$= \frac{P^2a^2b^2}{6EIL^2}(a+b)$$

$$= \frac{P^2a^2b^2}{6EIL}, \text{ Since, } a + b = L$$

$$\Delta_C = \frac{\delta U}{\delta P} = \frac{Pa^2b^2}{3EIL}$$

Example 3.12 Determine the vertical deflection at the free end and rotation at A in the overhanging beam shown in Figure 3.23(a). Assume constant EI. Use Castigliano's method.

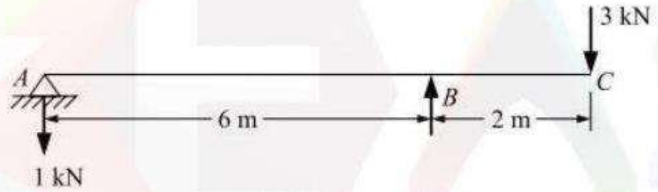


Figure 3.23(a) Example 3.12

Solution (1) Deflection at C: Taking 3 kN force as p,

$$R_B \times 6 = P \times 8$$

$$R_B = \frac{4}{3}P\uparrow$$

$$\therefore \quad R_A = \frac{P}{3}\downarrow$$

Figure 3.23(b) Reaction if 3 kN load is taken as ?

Bending moment expressions are noted, in the tabular form.

Table 3.10 Calculation table for Example 3.12

Portion	*AB*	*BC*
Origin	A	C
Limit	0–6	0–2
M	$\frac{-P}{3}x$	$-Px$
Flexural Rigidity	EI	EI

$$U = \int \frac{M^2}{2EI}dx$$

$$= \int_0^6 \frac{P^2x^2}{9} \times \frac{1}{2EI}dx + \int_0^2 \frac{P^2x^2}{2EI}dx$$

$$= \frac{P^2}{18EI}\left[\frac{x^3}{3}\right]_0^6 + \left[\frac{P^2x^3}{6EI}\right]_0^2$$

$$= \frac{4P^2}{EI} + \frac{4}{3} \times \frac{P^2}{EI}$$

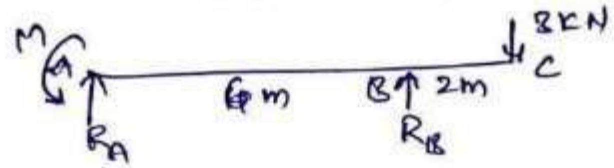
$$= \frac{5.333P^2}{EI}$$

$$\Delta_C = \frac{dU}{dP} = \frac{10.667P}{EI}$$

Substituting $P = 3$ kN, we get

$$\Delta_C = \frac{32}{EI}$$

For rotation at 'A', apply a dummy moment at 'A'.



$$\Sigma M_B = 0 \Rightarrow R_A \times 6 - M + 6 = 0$$

$$R_A = \frac{M-6}{6}$$

Portion	AB	BC
origin	A	C
Limit	0–6	0–2
M	$\left(\frac{M-6}{6}\right)x - M$	$3x$

Total strain Energy $U = \int_0^6 \left[\left(\frac{M-6}{6}\right)x - M\right]^2 \frac{1}{2EI}\,dx + \int_0^6 \frac{(3x)^2}{2EI}\,dx$

$$\theta_A = \frac{\partial U}{\partial M} = \int_0^6 2\left[\left(\frac{M-6}{6}\right)x - M\right]\left(\frac{x}{6} - 1\right)\frac{1}{2EI}\,dx + 0$$

Put $M = 0$ ($\because$ dummy)

~~$\theta = \int_0^6 (-x)\left(\frac{x}{6} - 1\right)\frac{1}{EI}\,dx + 0$~~

$$\theta = \int_0^6 (-x)\left(\frac{x}{6} - 1\right)\frac{1}{EI}\,dx$$

$$= \frac{1}{EI}\int_0^6 \left(\frac{-x^2}{6} + x\right)dx$$

$$= \frac{1}{EI}\left[\frac{1}{6}\left(\frac{-x^3}{3}\right)_0^6 + \left(\frac{x^2}{2}\right)_0^6\right]$$

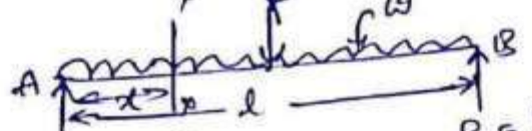
$$= \frac{1}{EI}\left[\frac{1}{6 \times 3}(-6)^3 + \frac{6^2}{2}\right]$$

$$= \frac{1}{EI}\left[\frac{-36}{3} + \frac{36}{2}\right]$$

$$\boxed{\theta_A = \frac{6}{EI}}$$

Calculate the central deflection & slope at ends of SSB carrying UDL w per unit length over the whole span.

Sol:- Central deflection



$R_A = \frac{wl+P}{2}$ $\qquad R_B = \frac{wl+P}{2}$

due to symmetry $R_A = R_B = \frac{wl+P}{2}$

~~Bending moment at section x-x = $\int \frac{M^2}{2EI} dx$~~

~~$U = 2\int_0^{l/2} \left(\frac{wx+P}{2}\right)^2$~~

Bending moment at section x-x $= \left(\frac{wl+P}{2}\right)x - \frac{wx^2}{2}$

Total strain energy stored $= \int \frac{M^2}{2EI} dx$

$$U = 2\int_0^{l/2} \left[\left(\frac{wl+P}{2}\right)x - \frac{wx^2}{2}\right]^2 \times \frac{1}{2EI} dx$$

$$U = \frac{1}{EI}\int_0^{l/2} \left[\left(\frac{wl+P}{2}\right)x - \frac{wx^2}{2}\right]^2 dx$$

central deflection

$$\Delta_C = \frac{\partial U}{\partial P} = \frac{1}{EI}\int_0^{l/2} 2\left[\left(\frac{wl+P}{2}\right)x - \frac{wx^2}{2}\right] \cdot \frac{x}{2} dx$$

$$\Delta_C = \frac{\partial U}{\partial P} = \frac{1}{EI}\int_0^{l/2} \left(\frac{wlx}{2} - \frac{wx^2}{2}\right) \cdot x \cdot dx \quad (\because P = 0)$$

$$\Delta_C = \frac{1}{EI}\int_0^{l/2} \left(\frac{wlx^2}{2} - \frac{wx^3}{2}\right) dx$$

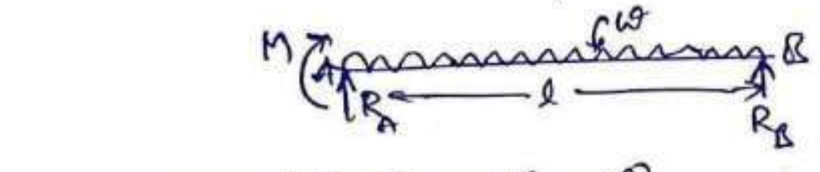
$$= \frac{1}{EI}\left[\frac{wl}{2}\left(\frac{x^3}{3}\right)_0^{l/2} - \frac{w}{2}\left(\frac{x^4}{4}\right)_0^{l/2}\right]$$

$$= \frac{1}{EI}\left[\frac{wl}{6}\left(\frac{l^3}{8}\right) - \frac{w}{8}\left(\frac{l^4}{16}\right)\right]$$

$$= \frac{1}{EI}\left(\frac{wl^4}{48} - \frac{wl^4}{128}\right)$$

$$\boxed{\Delta_C = \frac{5\,wl^4}{384\,EI}}$$

To find rotation at 'A', apply dummy moment at 'A'.



$\Sigma V = 0 \Rightarrow R_A + R_B = wl \rightarrow (1)$

$\Sigma M_A = 0 \Rightarrow -R_B l + wl \cdot \frac{l}{2} + M = 0$

$R_B l = \frac{wl^2}{2} + M$

$R_B = \frac{wl^2}{2l} + \frac{M}{l}$

$R_B = \frac{wl}{2} + \frac{M}{l}$

From eq (1)

$R_A = wl - R_B$

$= wl - \frac{wl}{2} - \frac{M}{l}$

$R_A = \frac{wl}{2} - \frac{M}{l}$

~~Total Strain Energy stored $U = \int_0^l \left(\frac{wl}{2} + \frac{M}{l}\right)^2 \frac{1}{2EI} dx$~~

~~$\theta_A = \frac{\partial U}{\partial M} = \int_0^l 2\left(\frac{wl}{2} + \frac{M}{l}\right)x$~~

Bending Moment section x-x $= +R_B x = -\left(\frac{wl}{2} + \frac{M}{l}\right)x + \frac{wx^2}{2}$ from support 'B'.

Total Strain Energy stored ~~$U = \int_0^l \left[-\left(\frac{wl}{2} + \frac{M}{l}\right)x\right]^2 \frac{1}{2EI} dx$~~

$\theta_A = \frac{\partial U}{\partial M} = \frac{1}{EI}\int_0^l M_x \frac{\partial M_x}{\partial M} dx \rightarrow (2)$

where $M_x = -\left(\frac{wl}{2} + \frac{M}{l}\right)x + \frac{wx^2}{2}$

$\frac{\partial U}{\partial M} = \frac{-x}{l}$

from eq (2) $\Rightarrow \theta_A = \frac{1}{EI}\int_0^l \left[-\left(\frac{wl}{2} + \frac{M}{l}\right)x + \frac{wx^2}{2}\right]\left(\frac{-x}{l}\right) dx$

$\theta_A = \frac{1}{EI}\int_0^l \left[\left(\frac{wl}{2} + \frac{M}{l}\right)x - \frac{wx^2}{2}\right]\left(\frac{x}{l}\right) dx$

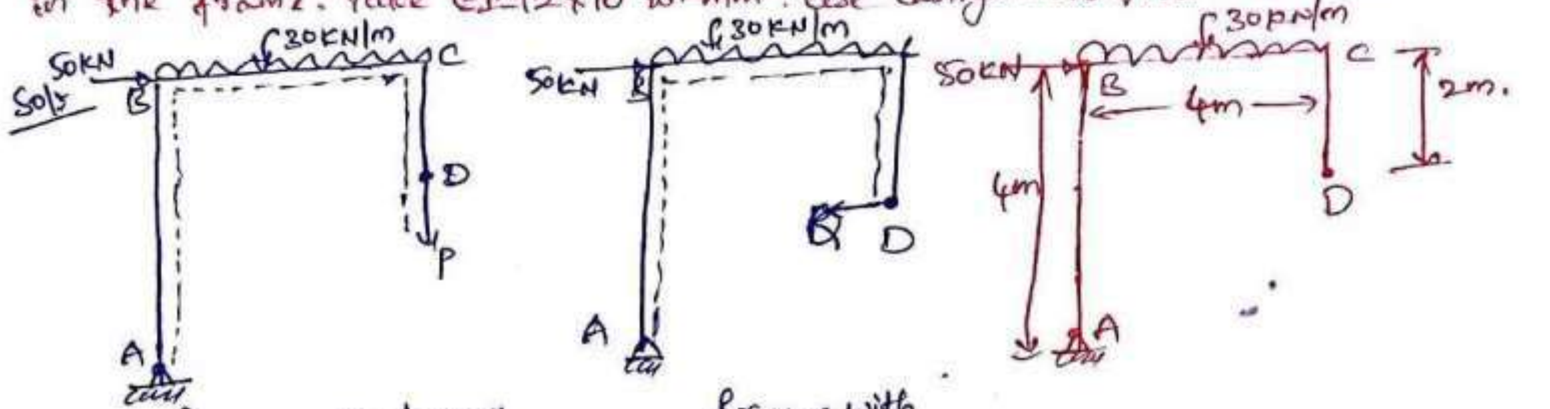
But $M = 0 \Rightarrow \theta = \frac{1}{EI}\int_0^l \left(\frac{wlx}{2} - \frac{wx^2}{2}\right)\left(\frac{x}{l}\right) dx$

$\theta_A = \frac{1}{EI}\int_0^l \left(\frac{wlx^2}{2l} - \frac{wx^3}{2l}\right) dx = \frac{1}{EI}\left(\frac{w}{2}\frac{x^3}{3} - \frac{wx^4}{2l(4)}\right)_0^l$

$\theta_A = \frac{1}{EI}\left(\frac{wl^3}{6} - \frac{wl^4}{8l}\right)_0^l = \frac{wl^3}{24EI}$ $\boxed{\therefore \theta_A = \frac{wl^3}{24EI}}$

Prob: Determine the vertical & horizontal displacement of the free end 'D' in the frame. Take $EI = 12\times10^{13}$ N-mm². Use Castigliano's first theorem. (10)

Sol:



frame with dummy vertical load 'P' at D

frame with dummy horizontal load 'P' at 'D'.

Vertical Deflection:-

Portion	Origin	Limit	Moment
AB	B	0-4	$-(4P+30\times4\times2+50x)$
BC	C	0-4	$-(Px+30x\cdot\frac{x}{2}) = -(Px+15x^2)$
CD	D	0-2	0

∵ 'P' is dummy load (P=0)

$$\text{strain Energy } U = \int_0^4 \frac{(4P+240+50x)^2}{2EI}dx + \int_0^4 \frac{(Px+15x^2)^2}{2EI}dx + 0$$

$$\Delta = \frac{\partial U}{\partial P} = \int_0^4 \frac{2(4P+240+50x)4}{2EI}dx + \int_0^4 \frac{2(Px+15x^2)x}{2EI}dx$$

'P' is dummy so, P=0.

$$\Delta = \int_0^4 \frac{4(240+50x)}{EI}dx + \int_0^4 \frac{15x^3}{EI}\cdot dx$$

$$= \frac{4}{EI}\left(240x + \frac{50x^2}{2}\right)_0^4 + \frac{15}{EI}\left(\frac{x^4}{4}\right)_0^4$$

$$= \frac{4}{EI}\left[240\times4 + \frac{50\times4^2}{2}\right] + \frac{15}{EI}\left(\frac{4^4}{4}\right)$$

$$= \frac{5440}{EI} + \frac{960}{EI} = \frac{6400}{EI} = \frac{6400}{12\times10^4}$$

$EI = 12\times10^{13}$ N-mm² $= \frac{12\times10^{13}}{10^3\times10^6} = 12\times10^4$ KN-m² $\Delta = 0.053$m

$\boxed{\Delta = 53.33 \text{ mm}}$

Horizontal deflection:

Portion	origin	limit	Moment
AB	B	0-4	$-[Q(2-x)+240+50x]$
BC	C	0-4	$-[2Q+15x^2]$
CD	D	0-2	Qx

Strain Energy $U = \int_0^4 \frac{[Q(2-x)+240+50x]^2}{2EI} dx + \int_0^4 \frac{(2Q+15x^2)^2}{2EI} dx + \int_0^2 \frac{(Qx)^2}{2EI} dx$

$$\Delta_{DH} = \frac{\partial U}{\partial Q} = \int_0^4 \frac{2[Q(2-x)+240+50x]}{2EI}(2-x)dx + \int_0^4 \frac{2(2Q+15x^2)}{2EI} 2dx + \int_0^2 \frac{2Qx}{2EI} x dx$$

substitute $Q=0$.

$$\Delta_{DH} = \int_0^4 \frac{(240+50x)(2-x)}{EI} + \int_0^4 \frac{30x^2}{EI} dx + 0$$

$$= \int_0^4 \frac{(480-240x+100x-50x^2)}{EI} + \int_0^4 \frac{30x^2}{EI} dx$$

$$= \int_0^4 \frac{(480-140x-50x^2)}{EI} + \int_0^4 \frac{30x^2}{EI} dx$$

$$= \frac{1}{EI}\left[480x - 140\frac{x^2}{2} - 50\frac{x^3}{3}\right]_0^4 + \frac{1}{EI}\left[\frac{30x^3}{3}\right]_0^4$$

$$= \frac{1}{EI}\left[480\times4 - 140\times\frac{4^2}{2} - 50\times\frac{4^3}{3}\right] + \frac{10}{EI}(4^3)$$

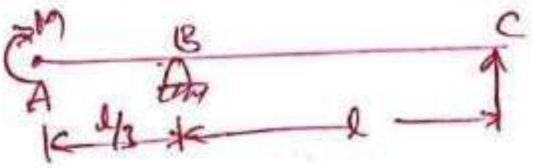
$$= \frac{1}{EI}\left[1920 - 1120 - \frac{3200}{3}\right] + \frac{640}{EI}$$

$$= \frac{-800}{3EI} + \frac{640}{EI} = \frac{373.33}{EI} = \frac{373.33}{12\times10^4}$$

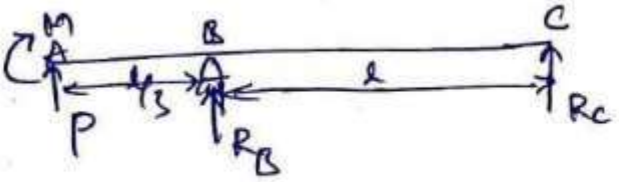
$$\Delta_{DH} = 0.00311\text{ m}$$

$$\Delta_{DH} = 3.11\text{ mm}$$

[Ex]:- Using Castigliano's first theorem, determine the deflection & rotation of the overhanging end 'A' of the beam loaded as shown in fig. (13)



Sol:- For vertical deflection:-



$$\Sigma V = 0 \Rightarrow R_B + R_C + P = 0 \longrightarrow (1)$$

$$\Sigma M_C = 0 \Rightarrow P\left(l + \frac{l}{3}\right) + R_B l + M = 0.$$

$$R_B l = -M - P\left(\frac{3l+l}{3}\right)$$

$$R_B = \frac{-M}{l} - \frac{4Pl}{3l} = \frac{-M}{l} - \frac{4P}{3}$$

$$R_B = -\left(\frac{M}{l} + \frac{4P}{3}\right) = \left(\frac{M}{l} + \frac{4P}{3}\right) \downarrow$$

From eq(1) $\Rightarrow$ $R_C = -P - R_B$

$$R_C = -P + \left(\frac{M}{l} + \frac{4P}{3}\right) = \left(\frac{M}{l} + \frac{P}{3}\right) \uparrow$$

Deflection of 'A'.

Then $\delta_A = \frac{\partial U}{\partial P} = \frac{1}{EI}\int_A^B M_x \cdot \frac{\partial M_x}{\partial M} \cdot dx + \frac{1}{EI}\int_B^C M_x \frac{\partial M_x}{\partial M} dx.$

For portion AB, $x = 0$ at 'A' & $x = \frac{l}{3}$ at B.

$$M_x = (-M - Px),$$

$$\frac{\partial M_x}{\partial P} = -x$$

For portion CB, $x = 0$ at 'C' & $x = l$ at B.

$$M_x = -R_C x = -\left(\frac{M}{l} + \frac{P}{3}\right)x$$

$$\frac{\partial M_x}{\partial P} = -\frac{x}{3}$$

Substituting above values

$$\delta_A = \frac{1}{EI}\int_0^{l/3} (M+Px)\cdot x \cdot dx + \frac{1}{EI}\int_0^{l}\left(\frac{M}{l}+\frac{Pl}{3}\right)x\left(\frac{x}{3}\right)\cdot dx$$

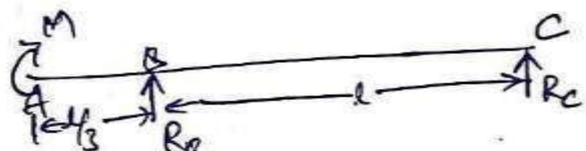
$$\delta_A = \frac{1}{EI}\int_0^{l/3} (Mx)\,dx + \frac{1}{EI}\int_0^{l}\frac{Mx^2}{3l}\,dx \quad (\because P=0)$$

$$\delta_A = \frac{1}{EI}\left(\frac{Mx^2}{2}\right)_0^{l/3} + \frac{1}{EI}\left(\frac{M}{3l}\left(\frac{x^3}{3}\right)\right)_0^{l}$$

$$\delta_A = \frac{1}{EI}\left[\frac{M}{2}\left(\frac{l^2}{9}\right)\right] + \frac{1}{EI}\left[\frac{Ml^3}{9l}\right] = \frac{Ml^2}{18EI} + \frac{Ml^2}{9EI}$$

$$\boxed{\delta_A = \frac{Ml^2}{6EI}}$$

Rotation at 'A':-



$\Sigma V = 0 \Rightarrow R_B + R_C = 0 \quad \rightarrow (1)$

$\Sigma M_B = 0 \Rightarrow M - R_C \times l = 0 \Rightarrow R_C l = M \Rightarrow R_C = \frac{M}{l} (\uparrow)$

from eq (1) $\Rightarrow R_B = -R_C = -\frac{M}{l} = \frac{M}{l} (\downarrow)$

Now Rotation at 'A', $\theta_A = \frac{\partial U}{\partial M} = \frac{1}{EI}\int_A^B M_x \cdot \frac{\partial M_x}{\partial M}dx + \frac{1}{EI}\int_C^B M_x \cdot \frac{\partial M_x}{\partial M}dx$

for portion AB, at A, $x=0$, at B $x = l/3$.

$$M_x = -M, \quad \frac{\partial M_x}{\partial M} = -1,$$

for portion CB: at C, $x=0$, at B, $x = l$.

$$M_x = -R_C x = -\frac{M}{l}x, \quad \frac{\partial M_x}{\partial M} = -\frac{x}{l}.$$

Substituting above values in the eq

$$\theta_A = \frac{1}{EI}\int_0^{l/3}(-M)(-1)dx + \frac{1}{EI}\int_0^{l}\left(-\frac{M}{l}x\right)\left(-\frac{x}{l}\right)dx$$

$$= \frac{M}{EI}(x)_0^{l/3} + \frac{M}{EI\cdot l^2}\left(\frac{x^3}{3}\right)_0^{l} = \frac{Ml}{3EI} + \frac{Ml}{3EI}$$

$$\boxed{\theta_A = \frac{2Ml}{3EI}} \text{ (Clockwise)}$$

Castigliano's Theorems:
The theorem of least work derives from what is known as Castigliano's second theorem. So, let's first state the two theorems of Carlo Alberto Castigliano (1847-1884) who was an Italian railroad engineer. In 1879, Castigliano published two theorems.

Castigliano's first theorem
The first partial derivative of the total internal energy (strain energy) in a structure with respect to any particular deflection component at a point is equal to the force applied at that point and in the direction corresponding to that deflection component.

This first theorem is applicable to linearly or nonlinearly elastic structures in which the temperature is constant and the supports are unyielding.

Castigliano's second theorem
The first partial derivative of the total internal energy in a structure with respect to the force applied at any point is equal to the deflection at the point of application of that force in the direction of its line of action.

The second theorem of Castigliano is applicable to linearly elastic (Hookean material) structures with constant temperature and unyielding supports.

Note that in the above statements, *force* may mean point force or couple (moment) and *displacement* may mean translation or angular rotation. Proofs of Castigliano's theorems are given at the end of this document.

Without further due, here is the theorem of least work, a.k.a. **Castigliano's theorem of least work:**

The redundant reaction components of a statically indeterminate structure are such that they make the internal work (strain energy) a minimum.

$$\therefore \text{ Total strain energy stored by the frame} = U = \sum \frac{S_1^2 l_1}{2A_1E}$$

$$= \sum (P_1 + XK_1)^2 \frac{l_1}{2A_1E}$$

$$\text{According to least work principle } \frac{\partial U}{\partial X} = 0$$

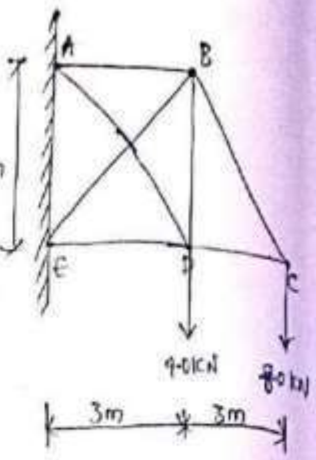
$$\Rightarrow \quad \sum 2(P_1 + XK_1)\frac{K_1 l_1}{2A_1E} = 0$$

$$\text{or,} \quad \sum \frac{P_1K_1l_1}{A_1E} + X\sum\frac{K_1^2 l_1}{A_1E} = 0$$

$$\text{or,} \quad X = -\frac{\sum \frac{P_1K_1l_1}{A_1E}}{\sum\frac{K_1^2 l_1}{A_1E}}$$

* Find the forces in the members of truss shown in figure. The cross area and young's modulus of all the members are the same.

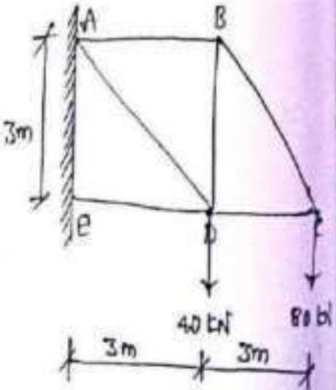
A) The vests



In this truss there is internal indeterminacy of 1 degree.

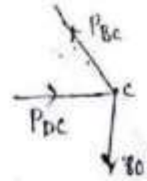
Force in the member BC is taken as the redundant force 'R'. The basic determinate structure with the given loading is shown in figure 1(A) and with the unit load in the direction of the redundant force is shown in figure 1(B)

* P-forces :-



1(A)

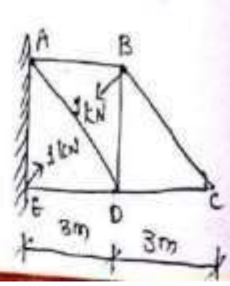
At joint C :-



$\Sigma V = 0$

$P_{BC} \sin 45^\circ = 80$

$P_{BC} = 80 \times \frac{1}{\sin 45^\circ}$



$= 80 \times \sqrt{2}$

$= 113.13$ kN [Tension]

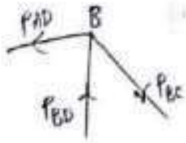
$\Sigma H = 0$

$P_{DC} = P_{BC} \cos 45^\circ$

$= 113.13 \times \frac{1}{\sqrt{2}}$

$= 80$ kN [Compression]

At joint B :-



$\Sigma V = 0$

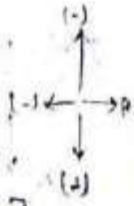
$P_{BD} = P_{BC} \sin 45^\circ$

$= 113.13 \times \frac{1}{\sqrt{2}}$

$P_{BD} = 80$ kN [Compression]

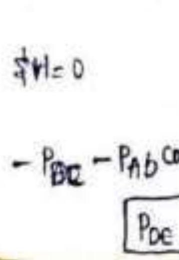
$\Sigma H = 0$

$P_{AB} = P_{BC} \cos 45^\circ = 113.13 \times \frac{1}{\sqrt{2}}$



$P_{AB} = 80$ kN [Tension]

At joint D :-



$\Sigma V = 0$

$40 + P_{BD} - P_{AD} \sin 45^\circ = 0$

$40 + 80 - P_{AD} \sin 45^\circ = 0$

$P_{AD} = 169.71$ kN (Tension)

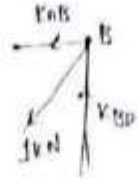
$\Sigma H = 0$

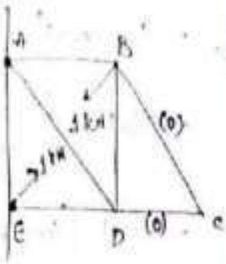
$-P_{DC} - P_{AD} \cos\theta + P_{DE} = 0$

$P_{DE} = 200$ kN [compression]

* K - forces :- [Removing external loads and applying unit load at which the deflection is to be determined]

joint B :-





$\Sigma V = 0$

$K_{BD} = 1 \sin 45^\circ$

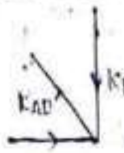
$= \frac{1}{\sqrt{2}} = 0.707$ [Compression]

$\Sigma H = 0$

$K_{AB} = 1 \cos 45^\circ$

$= 0.707$ [Compression]

Joint - D :-



$\Sigma V = 0$

$K_{BD} = K_{AD} \sin 45^\circ$

$K_{AD} = 1$ KN [Tension]

$\Sigma H = 0$

$K_{ED} = K_{AD} \cos 45^\circ$

$K_{ED} = \frac{1}{\sqrt{2}} =$

$K_{ED} = 0.707$ [Compression]

Member	P	K	l	PKl	$K^2 l$	S = P + RK
AB	- 80	0.707	3	- 169.68	1.49	- 192.3
BC	- 113.13	0	4.2	0	0	- 113.13
CD	80	0	3	0	0	80
DE	200	0.707	3	424.8	1.49	137.67
BD	80	0.707	3	169.68	1.49	17.67
AD	- 169.71	- 1	4.2	712.78	4.2	- 81.55
BE	0	1	4.2	0	4.2	88.14

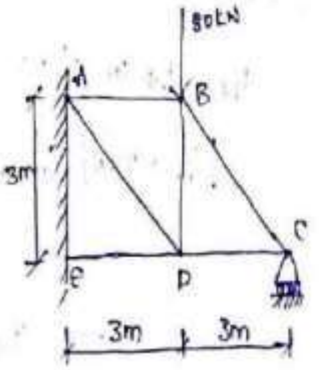
$\Sigma PKl = 1136.94$ $\Sigma K^2 l = 12.897$

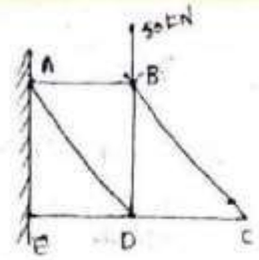
$$R = \frac{\Sigma \frac{PKl}{AE}}{\Sigma \frac{K^2 l}{AE}} = - \frac{1136.94}{12.897}$$

$= - 88.15$ KN.

* Find the forces in the members of the truss shown in figure. The cross sectional area and young's modulus of all the members are same.

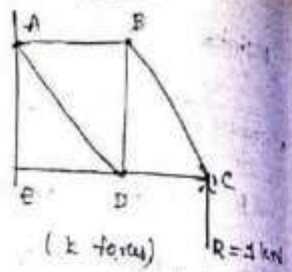
Ans)





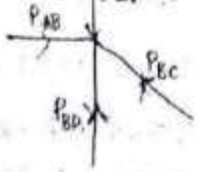
(P-forces)

R=4.0



(k forces)

At point B :-



$\Sigma V = 0$

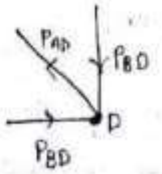
~~8 - P_{BC} sin θ = 0~~

$\boxed{80 = P_{BD}}$ [Compression]

$\Sigma H = 0$

$\boxed{P_{AB} = 0}$

At Joint D :-



$\Sigma V = 0$

$P_{BD} - P_{AD} \sin 45^\circ = 0$

$\boxed{P_{AD} = 113.137 \text{ kN}}$ [Tension]

$\Sigma H = 0$

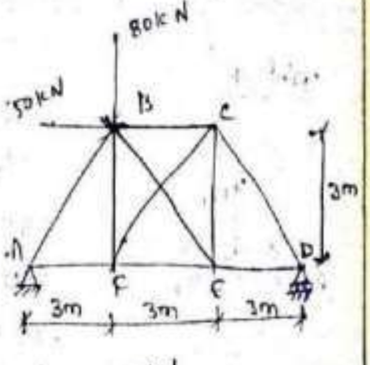
$P_{BD} - P_{AD} \cos 45^\circ = 0$

$\boxed{P_{BD} = 80\text{kN}}$ [Compression]

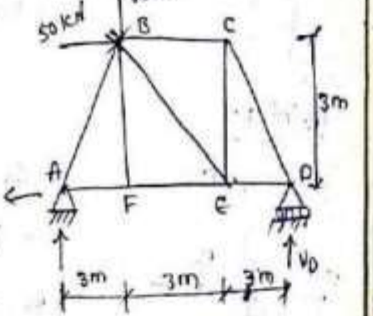
— Incomplete

* Find the forces in the members BE and CF as shown in figure. Assume same c/s area and young's modulus of for all members.

1) The structure is having 1 degree of internal indeterminacy. Let us consider, CF is redundant member.



P forces :-



Reactions :-

$\Sigma V = 0$:

$V_A + V_D = 80$ kN

$\Sigma H = 0$:

$H_A = 50$ kN

$\Sigma M_A = 0$:

$-V_D \times 9 + (8 \times 3) + (50 \times 3) = 0$

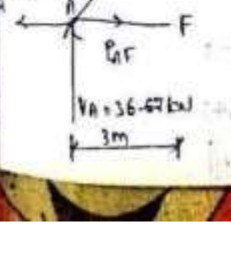
$-V_D \times 9 = +390$

$\boxed{V_D = 43.33 \text{ kN}}$

$V_A + 43.33 = 80 \Rightarrow V_A = 80 - 43.33$

$\boxed{V_A = 36.67\text{kN}}$

Joint A :-



$\Sigma V = 0$

$-V_A + P_{AB} \sin 45^\circ = 0$

$-36.67 = -P_{AB} \sin 45^\circ$

$P_{AB} = 36.67 \times \frac{1}{\sin 45}$

$\therefore \boxed{P_{AB} = 51.86 \text{ kN}}$ (Compression)

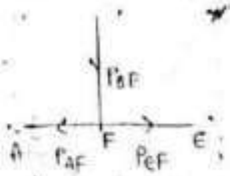
$\Sigma H = 0$

$-H_A + P_{AF} - P_{AB}\cos 45^\circ = 0$

$-50 + P_{AF} - 51.86\cos 45^\circ = 0$

$\boxed{P_{AF} = 86.67\ kN}$ [Tension]

Joint F :-



$\Sigma V = 0$

$P_{BF} = 0$

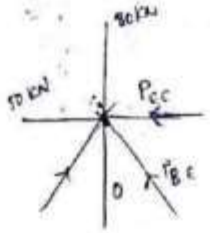
$\Sigma H = 0$

$-P_{AF} + P_{EF} = 0$

$-86.67 + P_{EF} = 0$

$\boxed{P_{EF} = 86.67 kN}$ [Tension]

Joint B:-



$\therefore \Sigma V = 0$

$-80 - P_{AB}\sin 45^\circ + P_{BE}\sin 45^\circ = 0$

$80 - 51.86\sin 45^\circ = P_{BE}\sin 45^\circ$

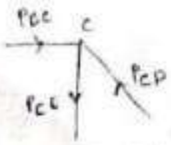
$\boxed{P_{BE} = 61.28\ kN}$ Compression

$\Sigma H = 0$

$50 - P_{BC} + P_{AB}\cos 45^\circ - P_{BE}\cos 45^\circ = 0$

$\boxed{P_{BC} = 43.34\ kN}$ [Compression]

* Joint C :-



$\Sigma V = 0$

$+P_{CE} - P_{CD}\sin 45^\circ = 0$ [Tension]

$P_{CE} = 61.28 \times \sin 45^\circ = 43.34\ kN$ comp

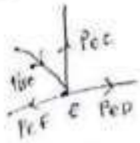
$\Sigma H = 0$

$P_{BC} - P_{CD}\cos 45^\circ = 0$

$43.34 - P_{CD}\cos 45^\circ$

$\boxed{P_{CD} = 61.29\ kN}$

Joint E :-



$\Sigma H = 0$

$-P_{EF} + P_{ED} + P_{BE}\cos 45^\circ = 0$

$-86.67 + P_{ED} + 61.28\cos 45^\circ = 0$

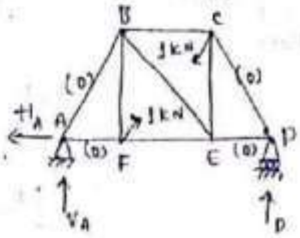
$\boxed{P_{ED} = 43.34\ kN}$ (Tension)

* K - forces :-

$\Sigma V = 0$

$V_A + V_D + 1\sin 45^\circ - 1\sin 45^\circ = 0$

$\boxed{V_A + V_D = 0}$



$\Sigma H = 0$

$-H_A + 1\cos 45^\circ - 1\cos 45^\circ = 0$

$\boxed{H_A = 0}$

$\Sigma M_A = 0$

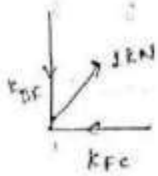
$-V_D \times 9 + [1\sin 45^\circ \times 6] - (1\sin 45^\circ \times 3) - (1\cos 45^\circ \times 3) = 0$

$\boxed{V_D = 0}$

$\boxed{V_A = 0}$

$\boxed{H_A = 0}$

* At joint F :-



$\Sigma V = 0$

$K_{BF} - 1\sin 45^\circ = 0$

$\boxed{K_{BF} = 0.707\ kN}$ [Compression]

$\Sigma H = 0$

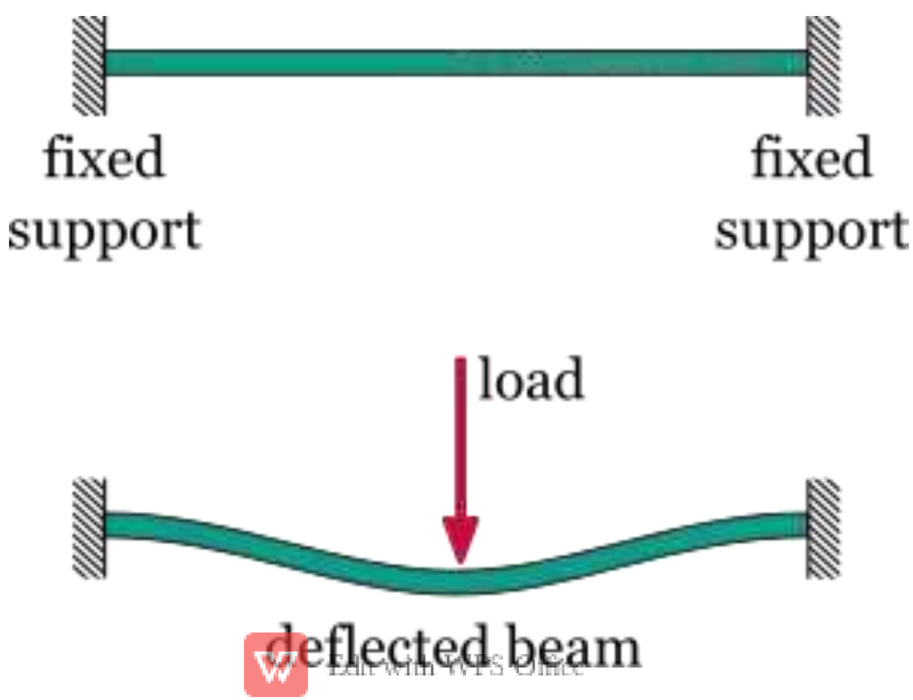
$-K_{FE} + 1\cos 45^\circ = 0$

$\boxed{K_{FE} = 0.707\ kN}$ [Compression]

FIXED BEAMS

FIXED BEAMS

- A beam whose both ends are fixed is known as a fixed beam. Fixed beam is also called as built-in or encaster beam.
- Incase of fixed beam both its ends are rigidly fixed and the slope and deflection at the fixed ends are zero



Advantages of fixed beams

(i) For the same loading, the maximum deflection of a fixed beam is less than that of a simply supported beam.

(ii) For the same loading, the fixed beam is subjected to lesser maximum bending moment.

(iii) The slope at both ends of a fixed beam is zero.

(iv) The beam is more stable and stronger.

Disadvantages of a fixed beam

(i) Large stresses are set up by temperature changes.

(ii) Special care has to be taken in aligning supports accurately at the same level.

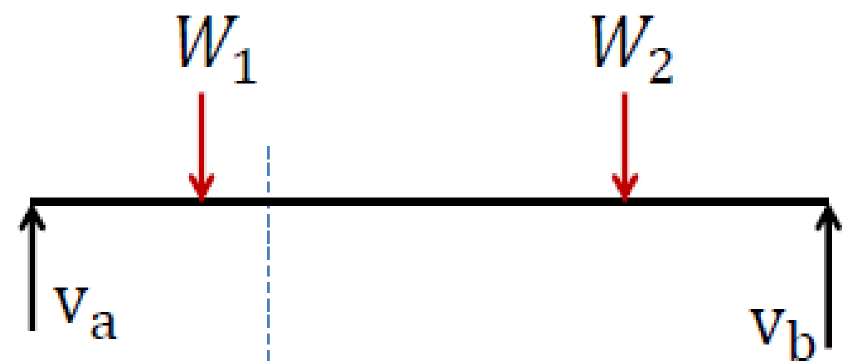
(iii) Large stresses are set if a little sinking of one support takes place.

(iv) Frequent fluctuations in loading render the degree of fixity at the ends very uncertain

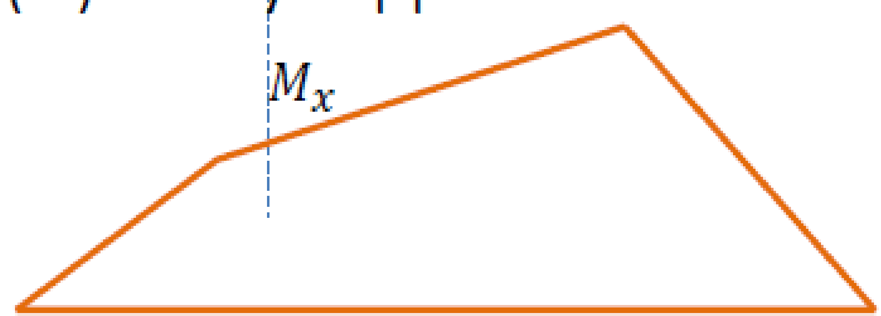
The beam may be analyzed in the following stages.

(i) Let us first consider the beam as Simply supported.

Let v_a and v_b be the vertical reactions at the supports A and B. Figure (ib) shows the bending moment diagram for this condition. At any section the bending moment M_x is a sagging moment.



(ia) Freely supported condition



(ib) Free B.M.D.

- (ii) Now let us consider the effect of end couples M_A and M_B alone.

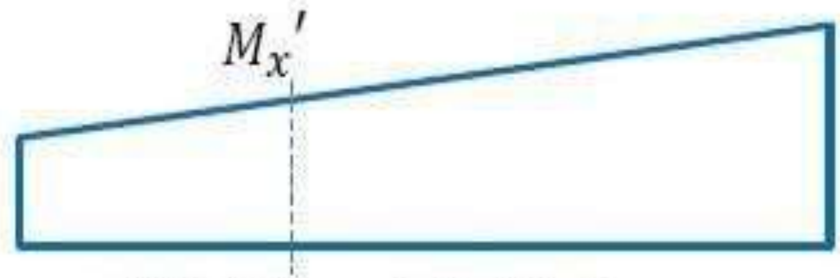
(iia) Effect of end couples

Let v be the reaction at each end due to this condition.

Suppose $M_B > M_A$.

Then $V = \frac{M_B - M_A}{L}$.

If $M_B > M_A$ the reaction V is upwards at B and downwards at A.



(iib) Fixed B.M.D.

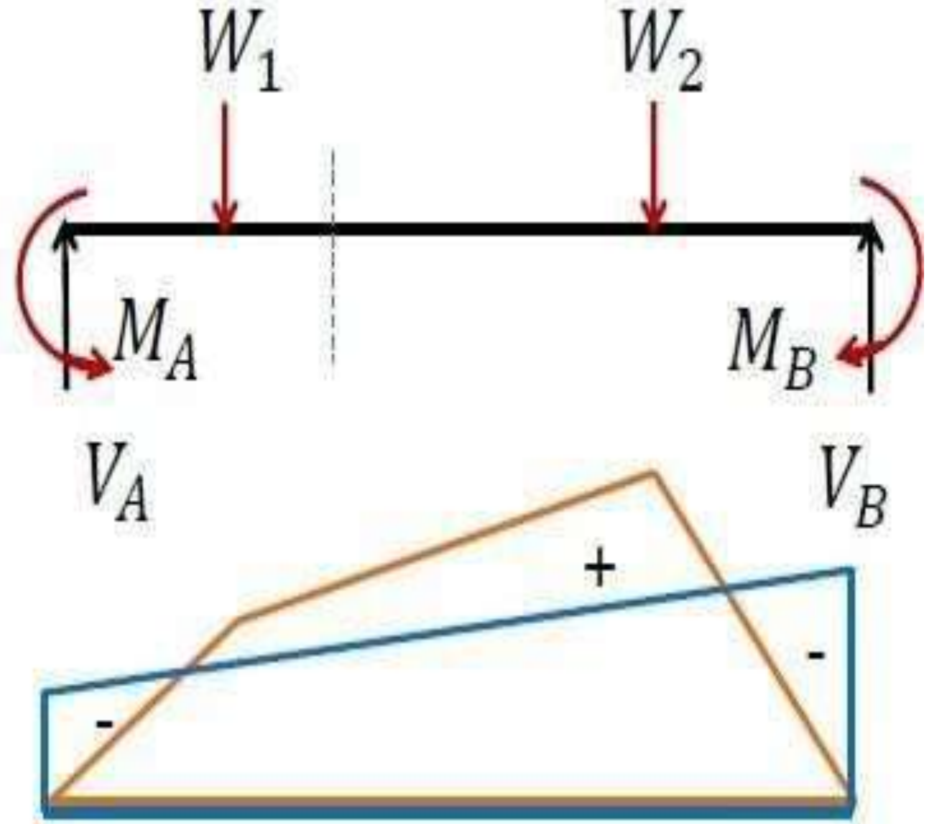
Fig (iib). Shows the bending moment diagram for this condition.

At any section the bending moment M_x' is hogging moment.

- Now the final bending moment diagram can be drawn by combining the above two B.M. diagrams as shown in Fig. (iiib)

Now the final reaction $V_A = v_a - v$

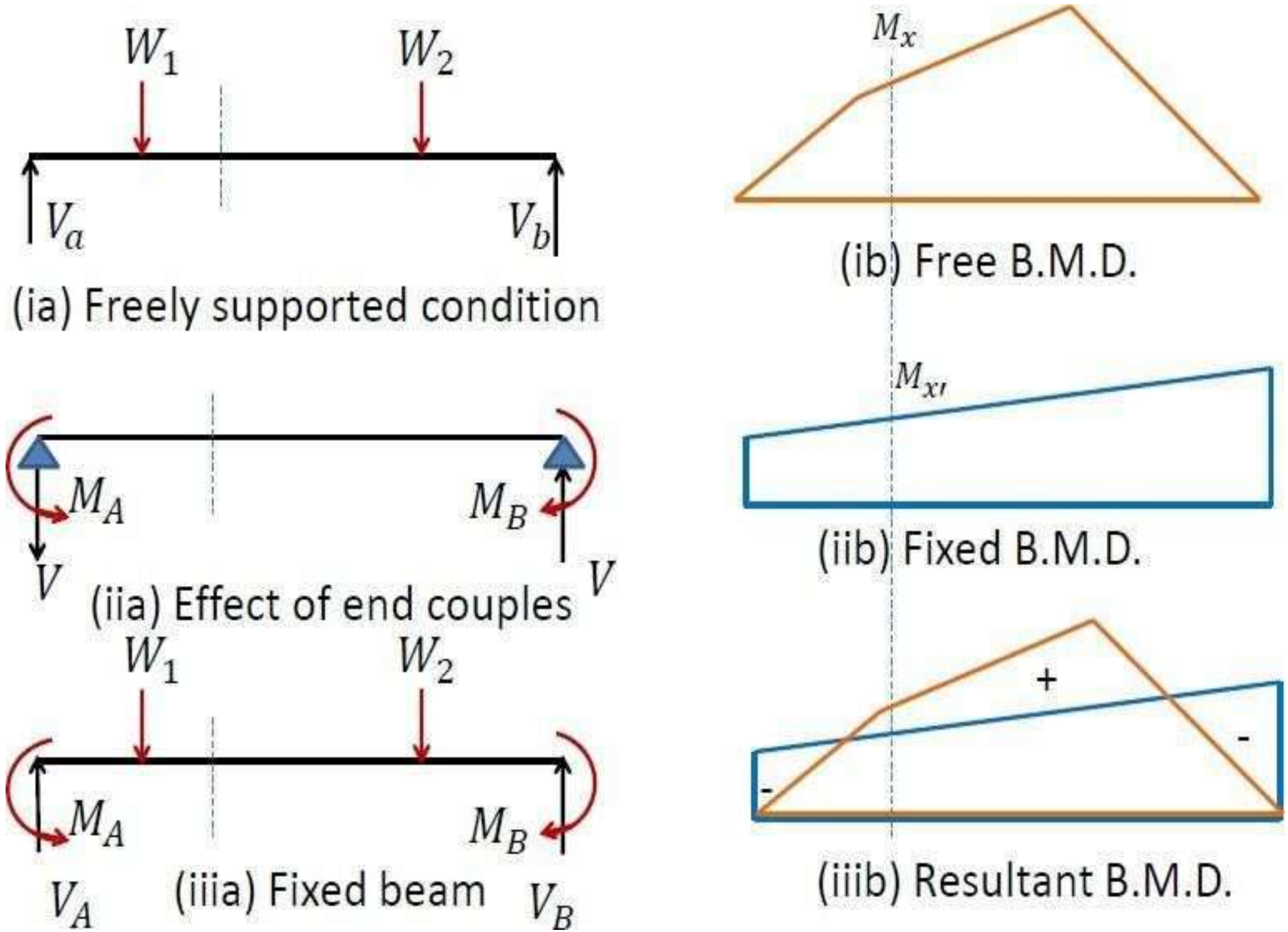
and $V_B = v_b + v$



(iiib) Resultant B.M.D.

The actual bending moment at any section X, distance x from the end A is given by,

$$EI \frac{d^2y}{dx^2} = M_x - M_x'$$



(ia) Freely supported condition

(ib) Free B.M.D.

(iia) Effect of end couples

(iib) Fixed B.M.D.

(iiia) Fixed beam

(iiib) Resultant B.M.D.

$$EI\frac{d^2y}{dx^2} = M_x - M_x'$$

- Integrating, we get,
- $EI\left[\frac{dy}{dx}\right]_0^l = \int_0^l M_x dx - \int_0^l M_x' dx$
- But at x=0, $\frac{dy}{dx} = 0$

  and at $x = l, \frac{dy}{dx} = 0$

Further $\int_0^l M_x dx$ = area of the Free BMD = a

$$\int_0^l M_x' dx = \text{area of the fixed B.M.D} = a'$$

Substituting in the above equation, we get,

$0 = a - a'$

$$\therefore a = a'$$

$$a = a'$$

∴ Area of the free B.M.D. = Area of the fixed B.M.D.

Again consider the relation,

$$EI\,\frac{d^2y}{dx^2} = M_x - M_x{}'$$

Multying by x we get,

$$EIx\,\frac{d^2y}{dx^2} = M_x x - M_x' x$$

- Integrating we get,
- $\int_0^l EIx\,\frac{d^2y}{dx^2} = \int_0^l M_x x\,dx - \int_0^l M_x' x\,dx$
- $\therefore EI\left[x\frac{dy}{dx} - y\right]_0^l$ = a$\bar{x}$-a′$\bar{x}'$
- Where $\bar{x}$= distance of the centroid of the free B.M.D. from A.

  and $\bar{x}'$= distance of the centroid of the fixed B.M.D. from A.

- Further at x=0, y=0 and $\frac{dy}{dx} = 0$
- and at x=l, y=0 and $\frac{dy}{dx} = 0.$
- Substituting in the above relation, we have

$$0 = a\bar{x} - a'\bar{x}^{+}$$

$$a\bar{x} = a'\bar{x}^{+}$$

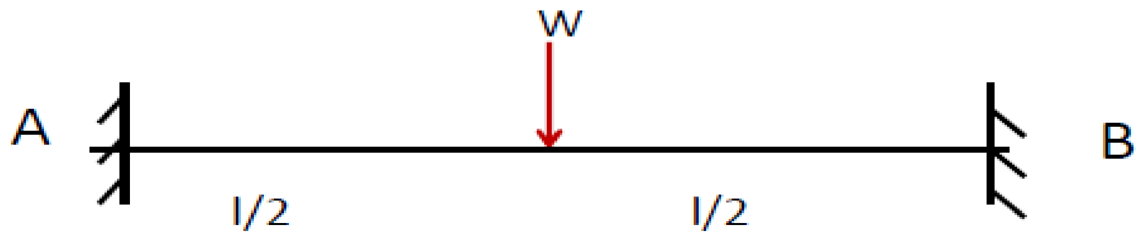
or $\bar{x} = \bar{x}$

∴ The distance of the centroid of the free B.M.D . From A= The distance of the centroid of the fixed B.M.D. from A.

$$\therefore a = a'$$

$$\bar{x} = \bar{x}$$

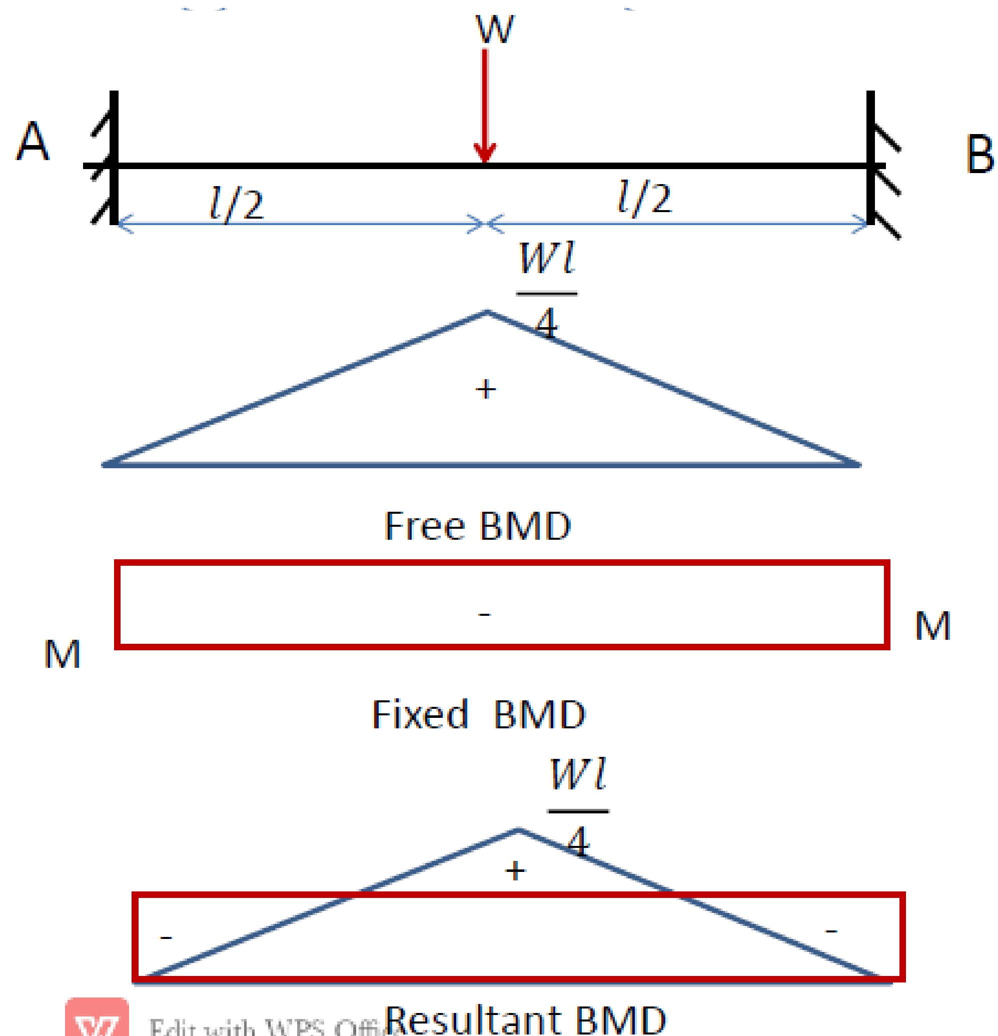
- Find the fixed end moments of a fixed beam subjected to a point load at the center.



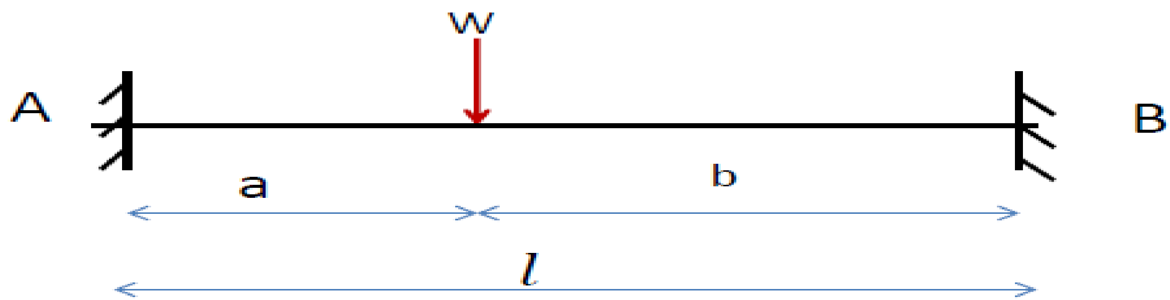
- $A' = A$

$$M \times l = \frac{1}{2} \times l \times \frac{Wl}{4}$$

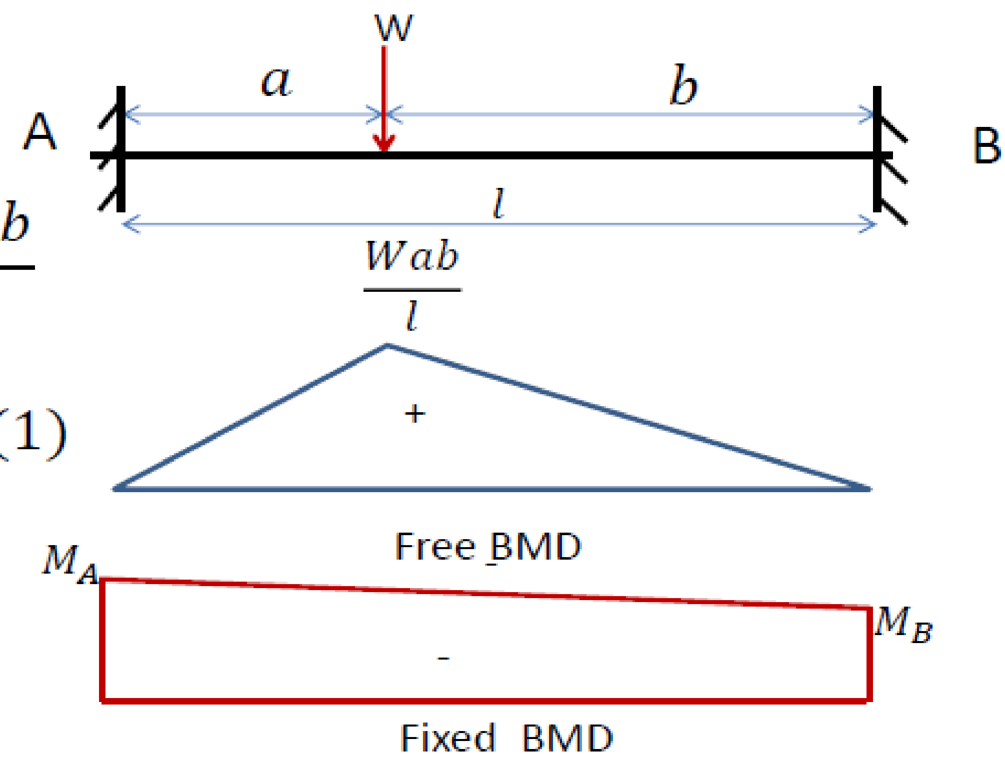
$$M = \frac{Wl}{8} = M_A = M_B$$



- Find the fixed end moments of a fixed beam subjected to a eccentric point load.



- $A' = A$



$$\frac{M_A + M_B}{2} \times l = \frac{1}{2} \times l \times \frac{Wab}{l}$$

$$M_A + M_B = \frac{Wab}{l} - - - - -(1)$$

- $x' = x$

$$\frac{M_A + 2M_B}{M_A + M_B} \times \frac{l}{3} = \frac{l + a}{3}$$

$$M_B = M_A \times \frac{a}{l - a}$$

$$M_B = \frac{a}{l} - - -(2)$$

$$M_A + M_B = \frac{Wab}{l} \text{-----}(1)$$

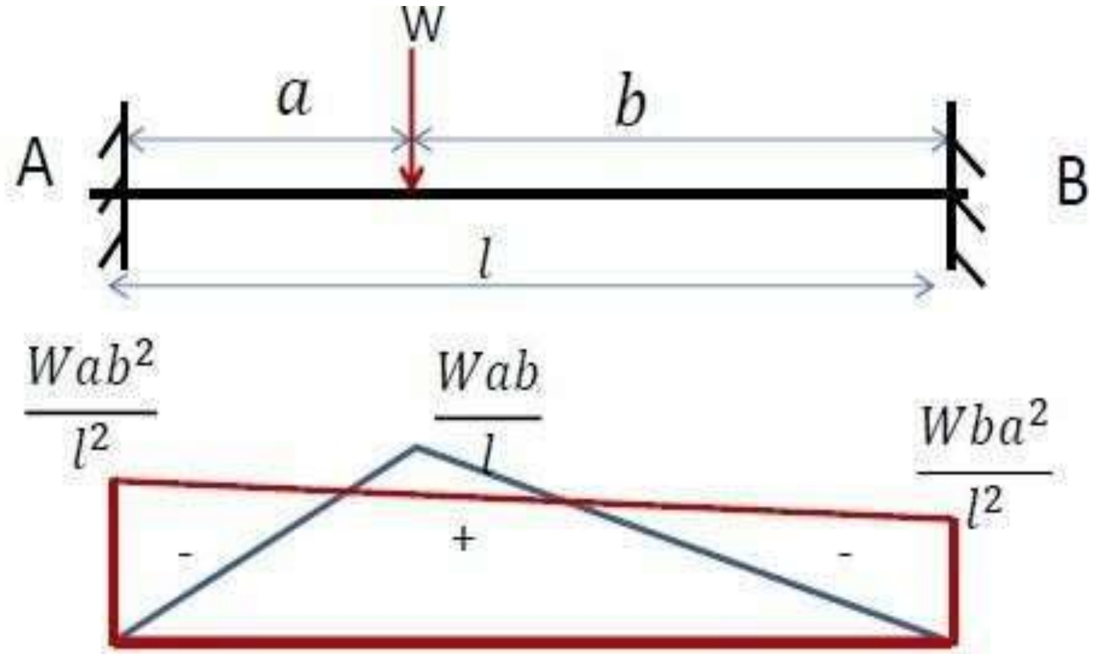
$$M_B = M_A \times \frac{a}{b} \text{---}(2)$$

By substituting (2) in (1),

$$M_A = \frac{Wab^2}{l^2}$$

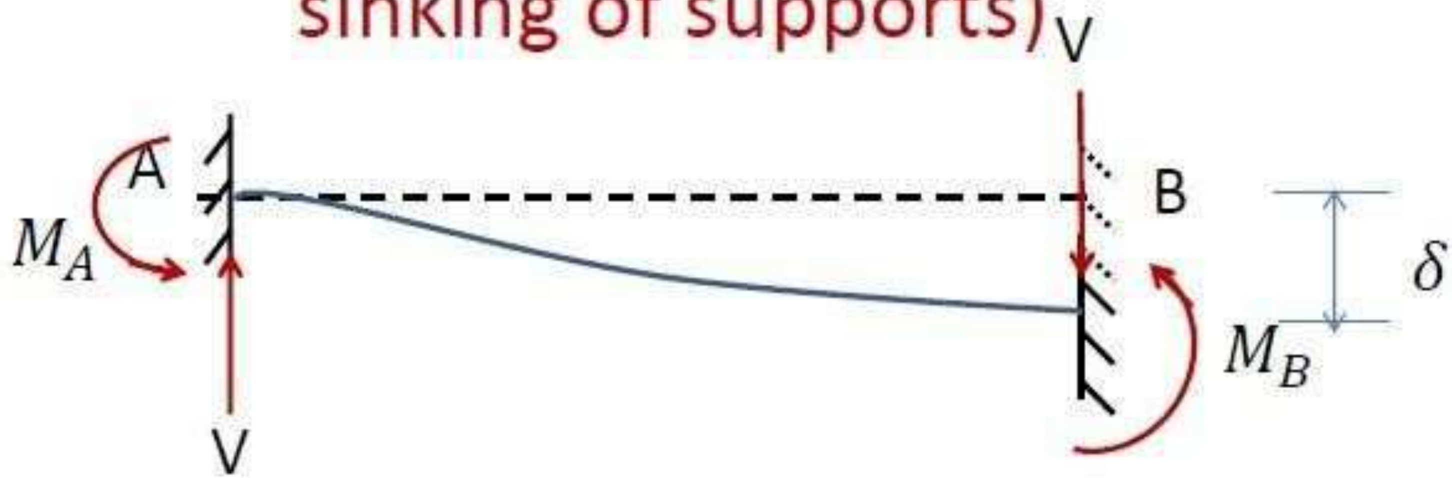
From (2),

$$M_B = \frac{Wba^2}{l^2}$$



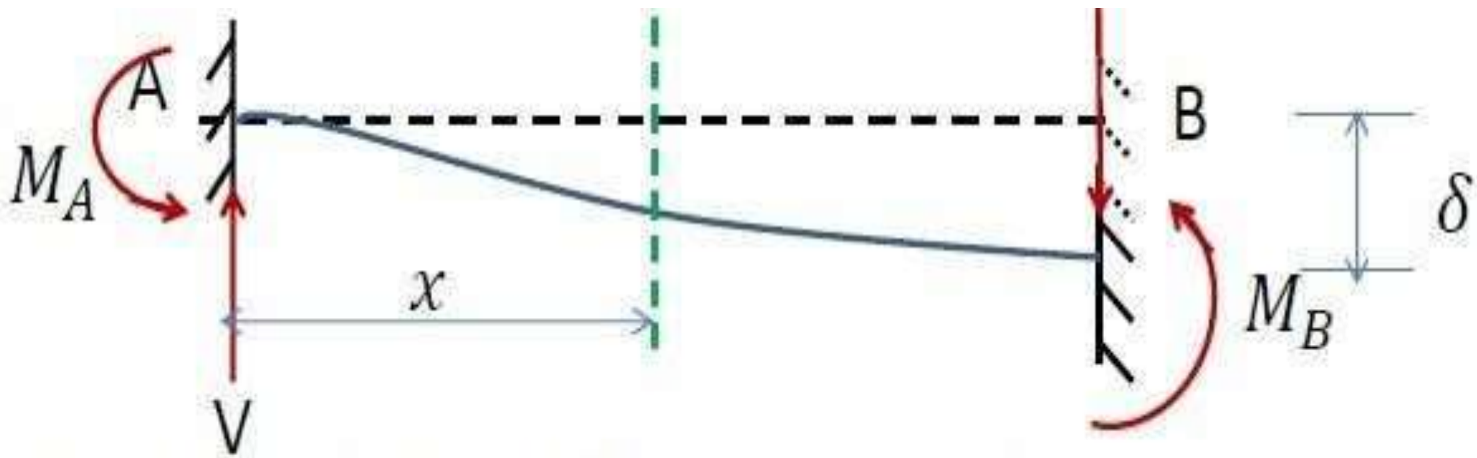
Resultant BMD

Fixed beam with ends at different levels (Effect of sinking of supports)



M_A is negative (hogging) and M_B is positive (sagging). Numerically M_A and M_B are equal.

Let V be the reaction at each support.



Consider any section distance x from the end A.

Since the rate of loading is zero, we have, with the usual notations

$$EI\frac{d^4y}{dx^4} = 0$$

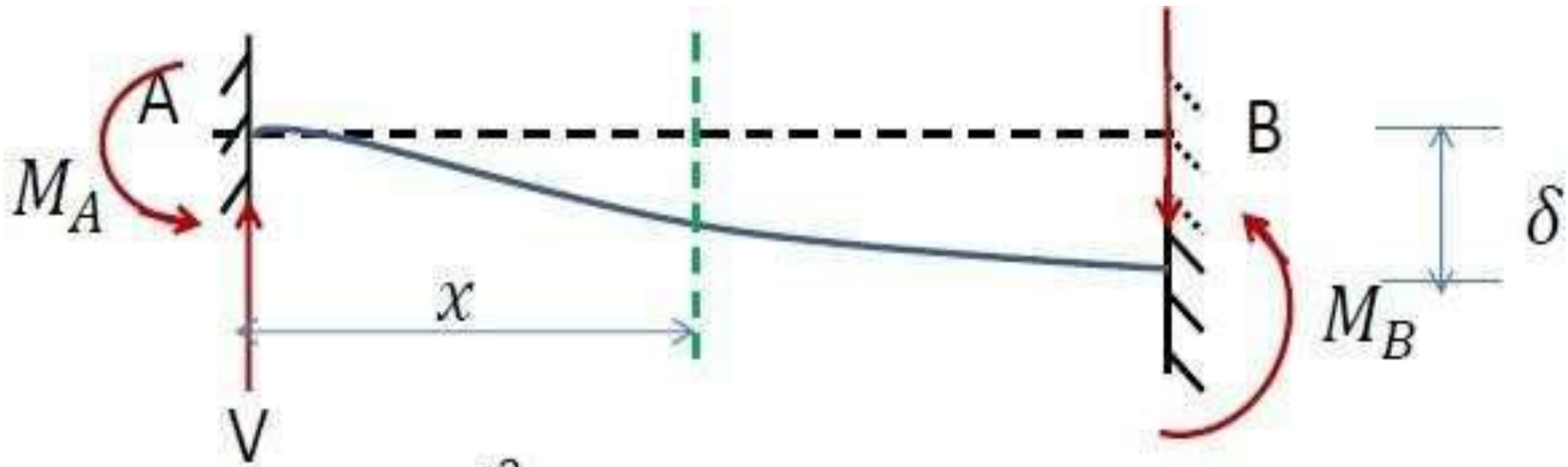
Integrating, we get,

Shear force = $EI\frac{d^3y}{dx^3} = C_1$

Where C_1 is a constant

At $x = 0$, $S.F. = +V$

$$\therefore C_1 = V$$



B.M. at any section = $EI\dfrac{d^2y}{dx^2} = Vx + C_1$

At $x = 0$, $B.M. = -M_A$

$$\therefore\ C_2 = -M_A$$

$$\therefore EI\frac{d^2y}{dx^2} = Vx - M_A$$

Integrating again,

$EI\dfrac{dy}{dx} = \dfrac{V}{2}x^2 - M_Ax + C_3$ (Slope equation)

But at $x = 0$, $\dfrac{dy}{dx} = 0 \qquad \therefore C_3 = 0$

Integrating again,

$$EI\,y = \frac{Vx^3}{6} - \frac{M_A x^2}{2} + C_4$$ ------ (Deflection equation)

But at $x = 0, y = 0$

$$\therefore C_4 = 0$$

At $x = l, y = -\delta$

$$-EI\,\delta = \frac{Vl^3}{6} - \frac{M_A l^2}{2} \text{ ---------- (i)}$$

But we also know that at B, $x = l\ and\ \frac{dy}{dx} = 0$

And substitute in slope Eq. $EI\frac{dy}{dx} = \frac{V}{2}x^2 - M_A x$

$$\therefore 0 = \frac{Vl^2}{2} - M_A l$$

$$\therefore V = \frac{2M_A}{l} \text{ --------- (ii)}$$

Substituting in deflection Eq.(i) i.e., $-EI\,\delta = \frac{Vl^3}{6} - \frac{M_A l^2}{2}$;we have,

$$-EI\,\delta = \frac{2M_A}{l} \times \frac{l^3}{6} - \frac{M_A l^2}{2}$$

$$EI\,\delta = \frac{M_A l^2}{6}$$

$$\therefore M_A = \frac{6EI\delta}{l^2}$$

Hence the law for the bending moment at any section distant x from A is given by,

$$M = EI\frac{d^2y}{dx^2} = \mathrm{V}x - M_A$$

$$\therefore M = \frac{2M_A}{l}x - \frac{6EI\delta}{l^2}$$

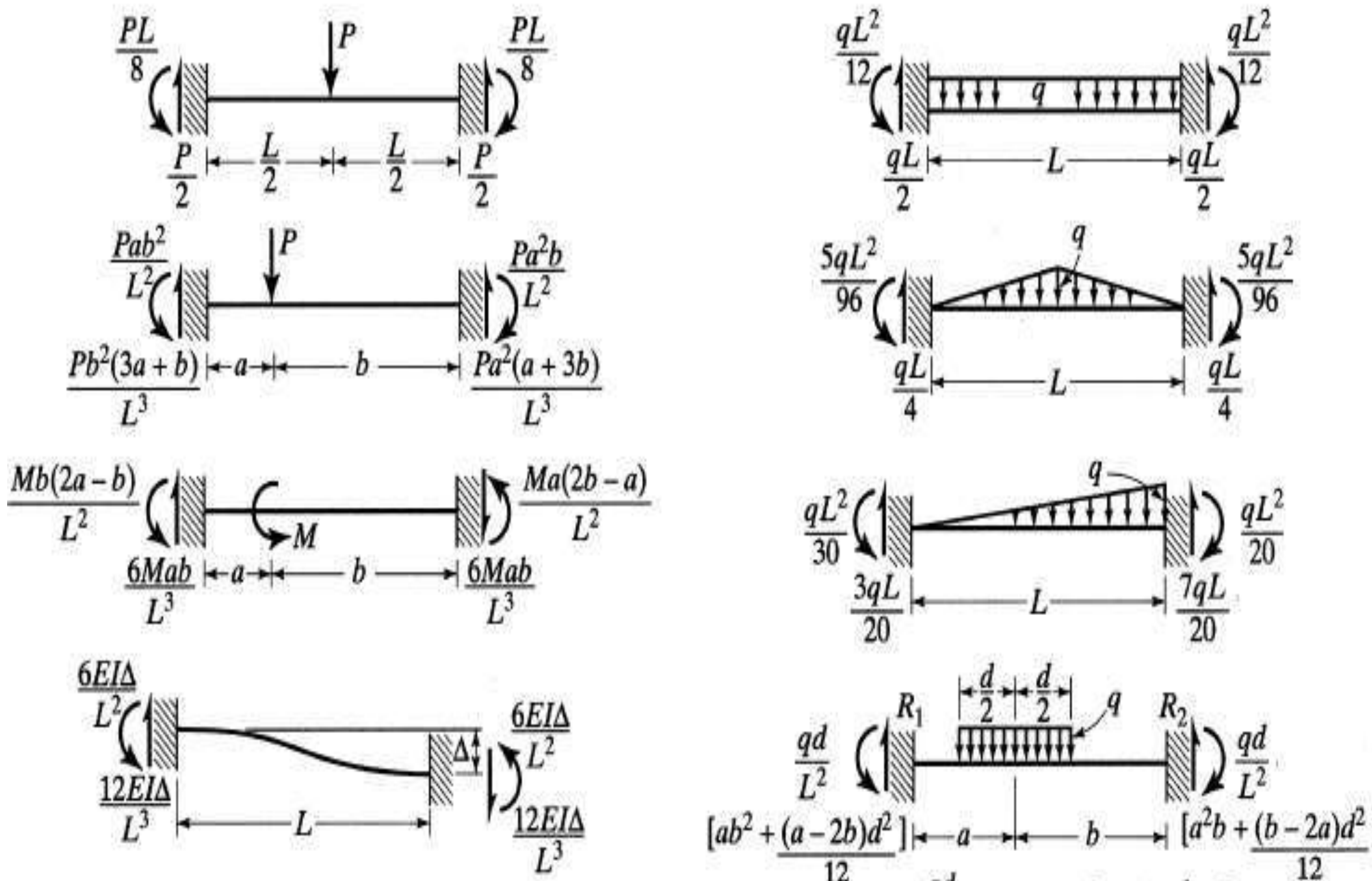
$$\text{But for B. M. at B, put x} = \text{l},$$

$$\therefore M_B = \frac{2M_A}{l} \times l - \frac{6EI\delta}{l^2} = \frac{12EI\delta}{l^2} - \frac{6EI\delta}{l^2} = \frac{6EI\delta}{l^2}$$

Hence when the ends of a fixed beam are at different levels,

The fixing moment at each end = $\frac{6EI\delta}{l^2}$ numerically.

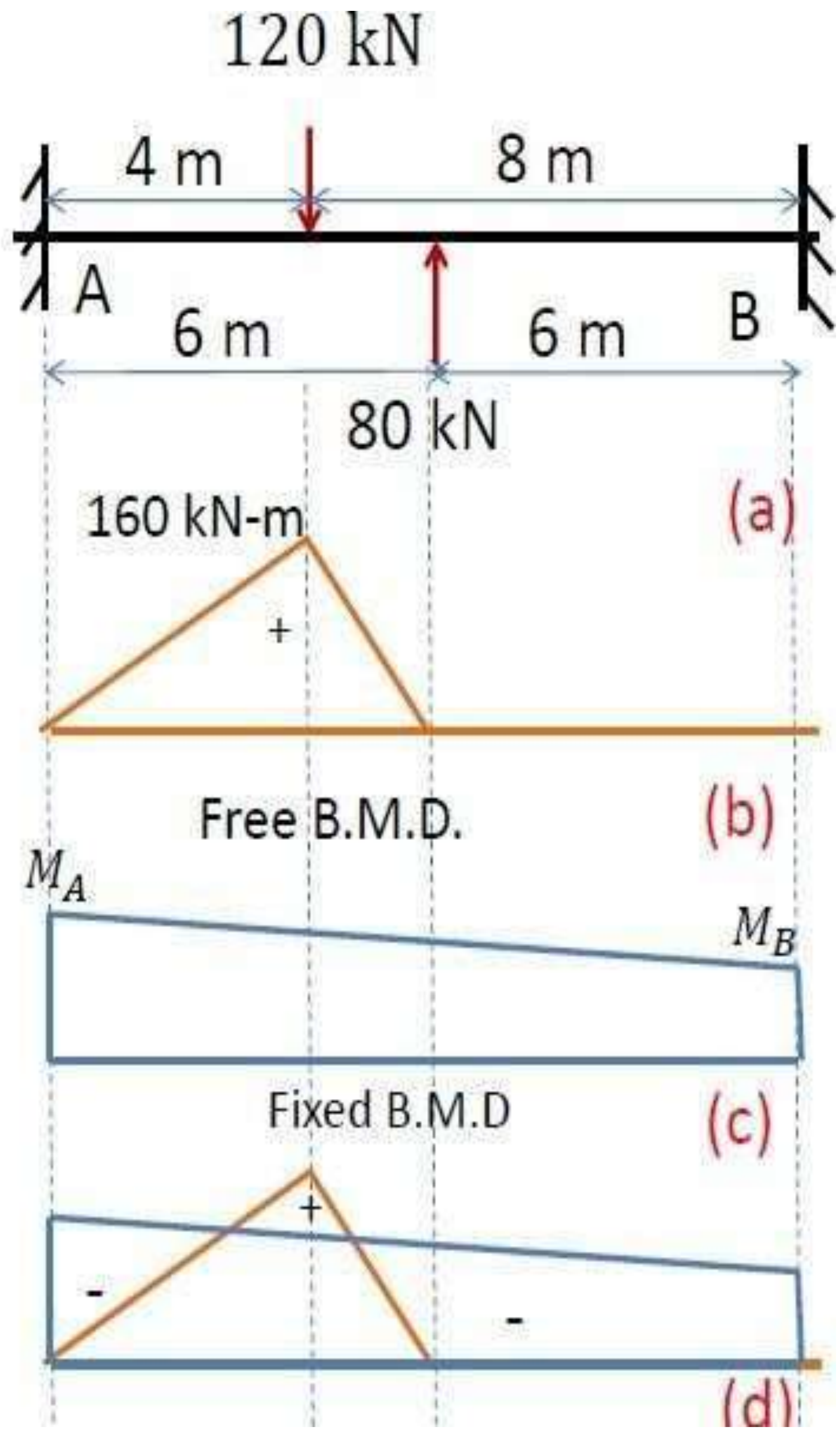
At the higher end this moment is a hogging moment and at the lower end this moment is a sagging moment.



$$R_1 = \frac{qd}{L^3}\left[(2a + L)b^2 + \left(\frac{a - b}{4}\right)d^2 \right]$$

$$R_2 = \frac{qd}{L^3}\left[(2b + L)a^2 - \left(\frac{a - b}{4}\right)d^2 \right]$$

- Solution:
- The M (Free B.M.) and M' (Fixed B.M.) diagrams have been shown in Fig.(b) and (c) respectively.



For the M-Diagram:

$$A = \frac{1}{2} \times 6 \times 160 = 480\ kNm$$

For the M' diagram:

$$A' = \frac{M_A + M_B}{2} \times 12 = 6(M_A + M_B)$$

- Area of the fixed B.M. D. = Area of the free B.M.D.

A' = A

$$6(M_A + M_B) = 480$$

$$M_A + M_B = 80 \text{-----------} (1)$$

The distance of the centroid of the free B.M. D. from A = The distance of the centroid of the fixed B.M.D. from A.

i.e., $x = x'$

$$\frac{6+4}{3} = \left(\frac{M_A + 2M_B}{M_A + M_B}\right) \times \frac{12}{3}$$

$$(M_A + 2M_B)12 = (M_A + M_B)10$$

$$12M_A + 24\, M_B - 10\, M_A - 10M_B = 0$$

$$2M_A + 14\, M_B = 0$$

$$M_A = -7M_B \text{ -----} (2)$$

- Substitute $M_A = -7M_B$ in equation (1)

$$-7M_B + M_B = 80$$

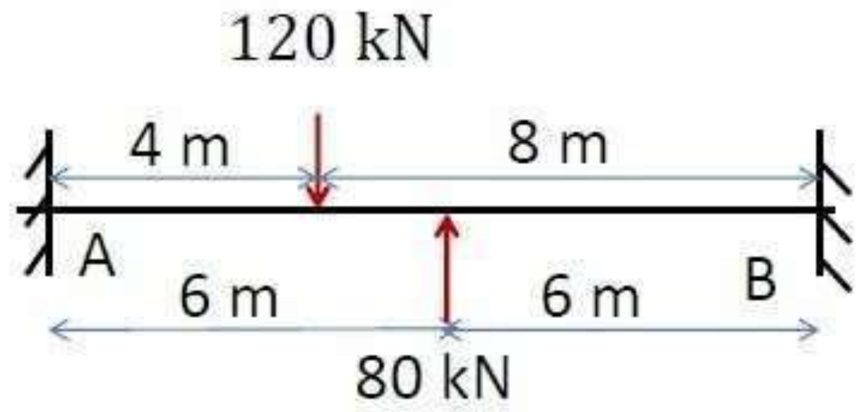
$$\therefore M_B = \frac{-80}{6} = -13.33$$

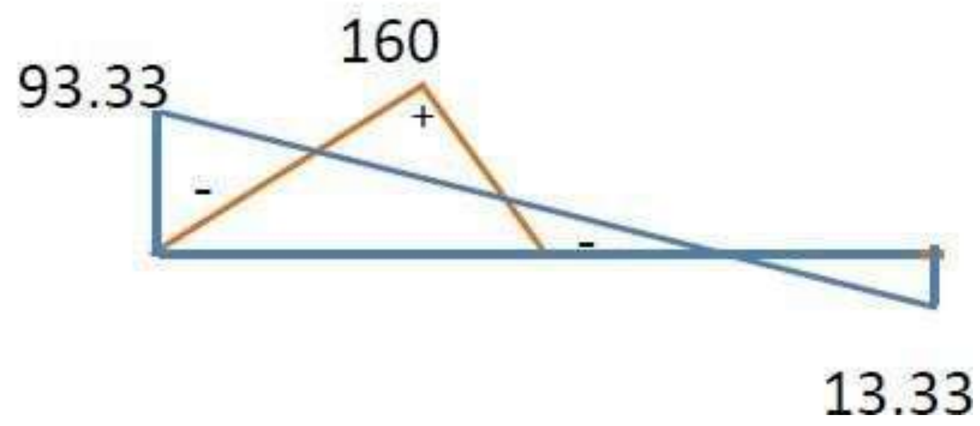
$$M_B = -13.33 \text{ kNm}$$

$$M_A = -7M_B$$

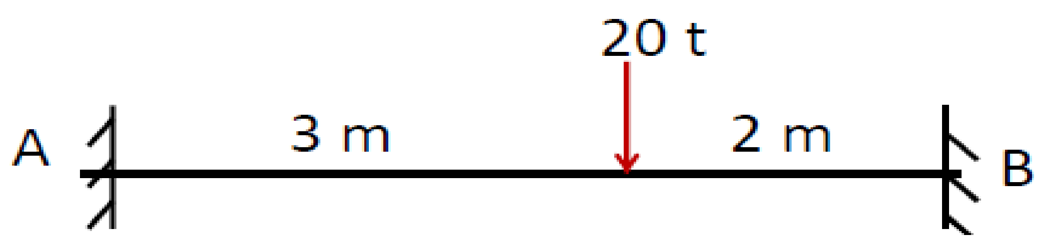
$$= -7(-13.33) = 93.33$$

$$\therefore M_A = 93.33\ kNm$$

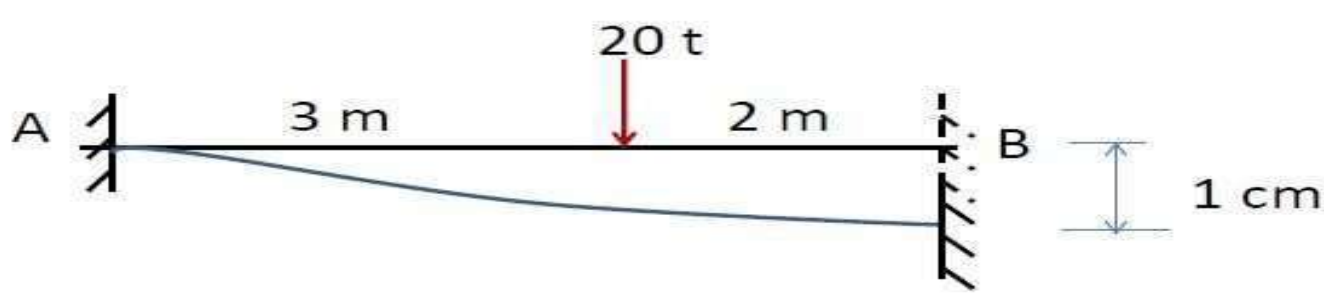




- A fixed beam of span 5 metres carries a concentrated load of 20 t at 3 meters from the left end. If the right end sinks by 1 cm, find the fixing moments at the supports. For the beam section take I=30,000 cm^4 and E=2x10^3 t/cm^2. Find also the reaction at the supports.

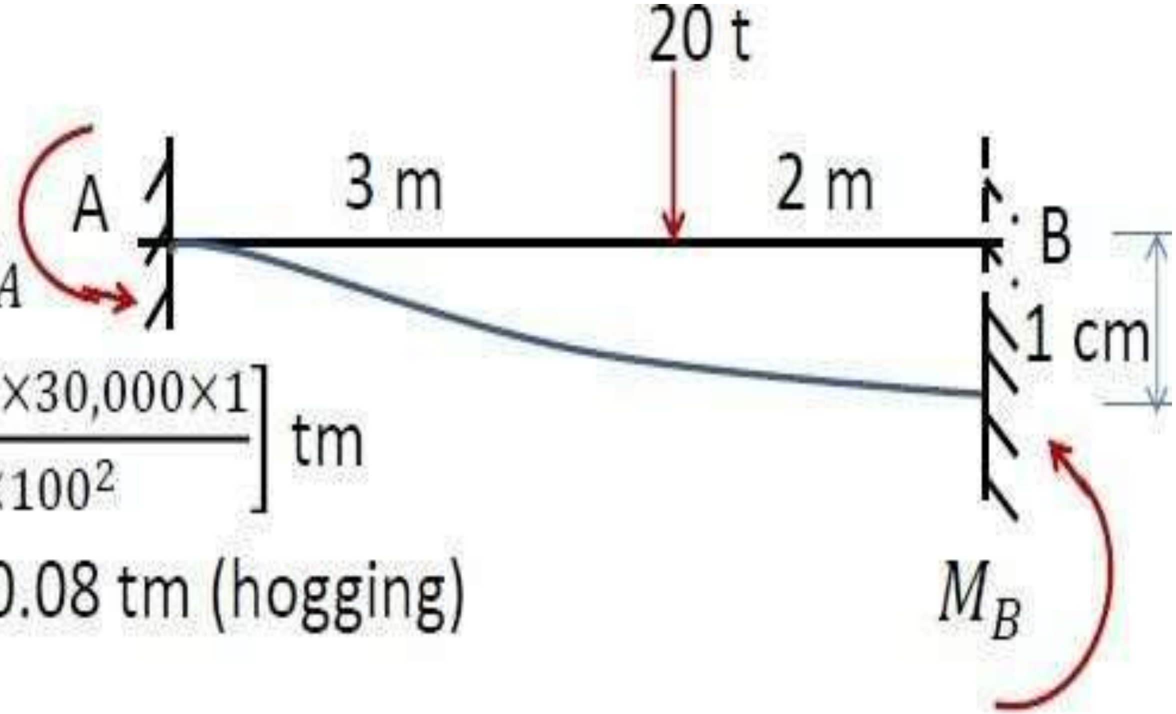


- A fixed beam of span 5 metres carries a concentrated load of 20 t at 3 meters from the left end.



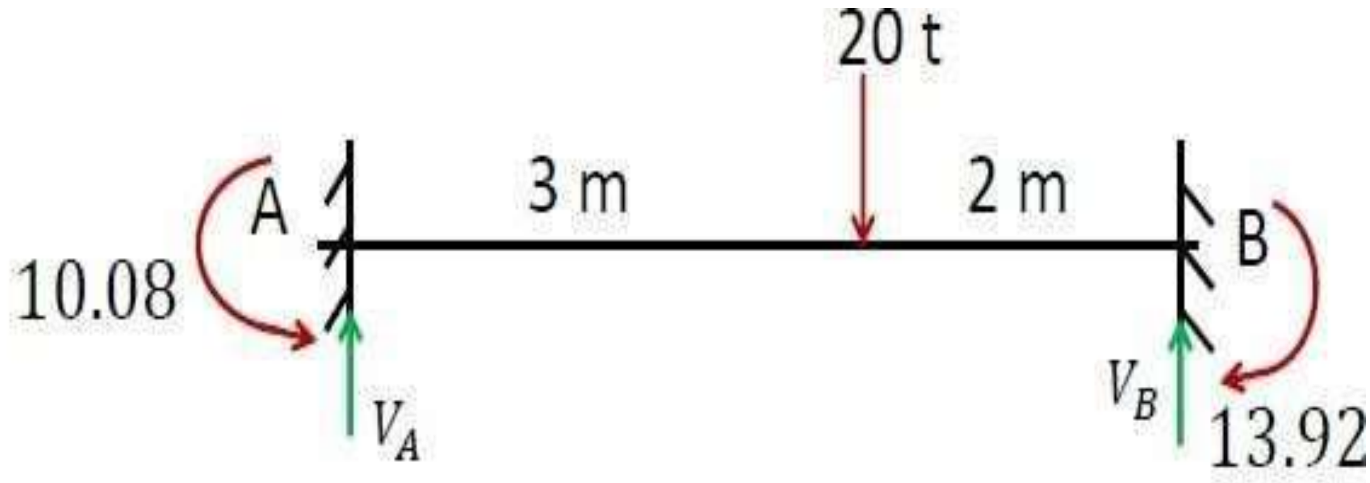
- The right end sinks by 1 cm, find the fixing moments at the supports.

- $M_A = -\frac{Wab^2}{l^2} - \frac{6EI\delta}{l^2}$



- $= -\left[\frac{20\times3\times2^2}{5^2} + \frac{6\times2\times10^3\times30{,}000\times1}{5^2\times100^2}\right]$ tm
- $= -[9.6 + 0.48]$ tm=-10.08 tm (hogging)

- $M_B = -\frac{Wba^2}{l^2} + \frac{6EI\delta}{l^2}$
- $= \left[-\frac{20\times2\times3^2}{5^2} + \frac{6\times2\times10^3\times30{,}000\times1}{5^2\times100^2}\right]$ tm
- $= [-14.4 + 0.48]$ tm= -13.92 tm (hogging)



- **Reaction at A:**
- $\sum M_B = 0,$
- $V_A \times 5 + 13.92 - 10.08 - (20 \times 2) = 0$
- $\therefore V_A = 7.232$ t
- **Reaction at B:**
- $\therefore V_B = 20 - 7.232 = 12.768$ t.

Continuous Beams

Introduction:

- Beams are made continuous over the supports to increase structural integrity.
- A continuous beam provides an alternate load path in the case of failure at a section.
- In regions with high seismic risk, continuous beams and frames are preferred in buildings and bridges.
- A continuous beam is a statically indeterminate structure.
-

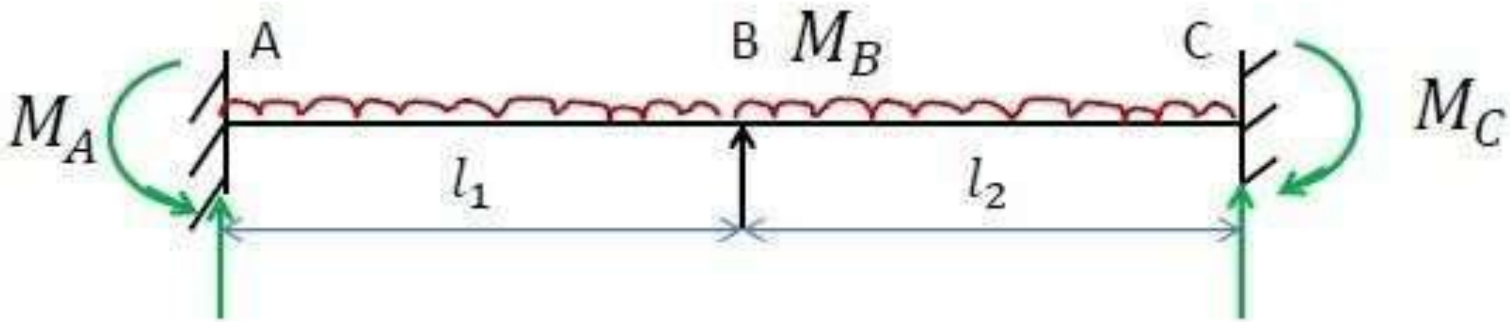
The advantages of a continuous beam as compared to a simply supported beam are as follows

1) For the same span and section, vertical load capacity is more.
2) Mid span deflection is less.
3) The depth at a section can be less than a simply supported beam for the same span. Else, for the same depth the span can be more than a simply supported beam.
   - The continuous beam is economical in material.
4) There is redundancy in load path.
   - Possibility of formation of hinges in case of an extreme event.
5) Requires less number of anchorages of tendons.
6) For bridges, the number of deck joints and bearings are reduced.
   - Reduced maintenance

There are of course several disadvantages of a continuous beam as compared to a simply supported beam.

1) Difficult analysis and design procedures.
2) Difficulties in construction, especially for precast members.
3) Increased frictional loss due to changes of curvature in the tendon profile.
4) Increased shortening of beam, leading to lateral force on the supporting columns.
5) Secondary stresses develop due to time dependent effects like creep and shrinkage, settlement of support and variation of temperature.
6) The concurrence of maximum moment and shear near the supports needs proper detailing of reinforcement.
7) Reversal of moments due to seismic force requires proper analysis and design.

Clapeyron's theorem of three moments



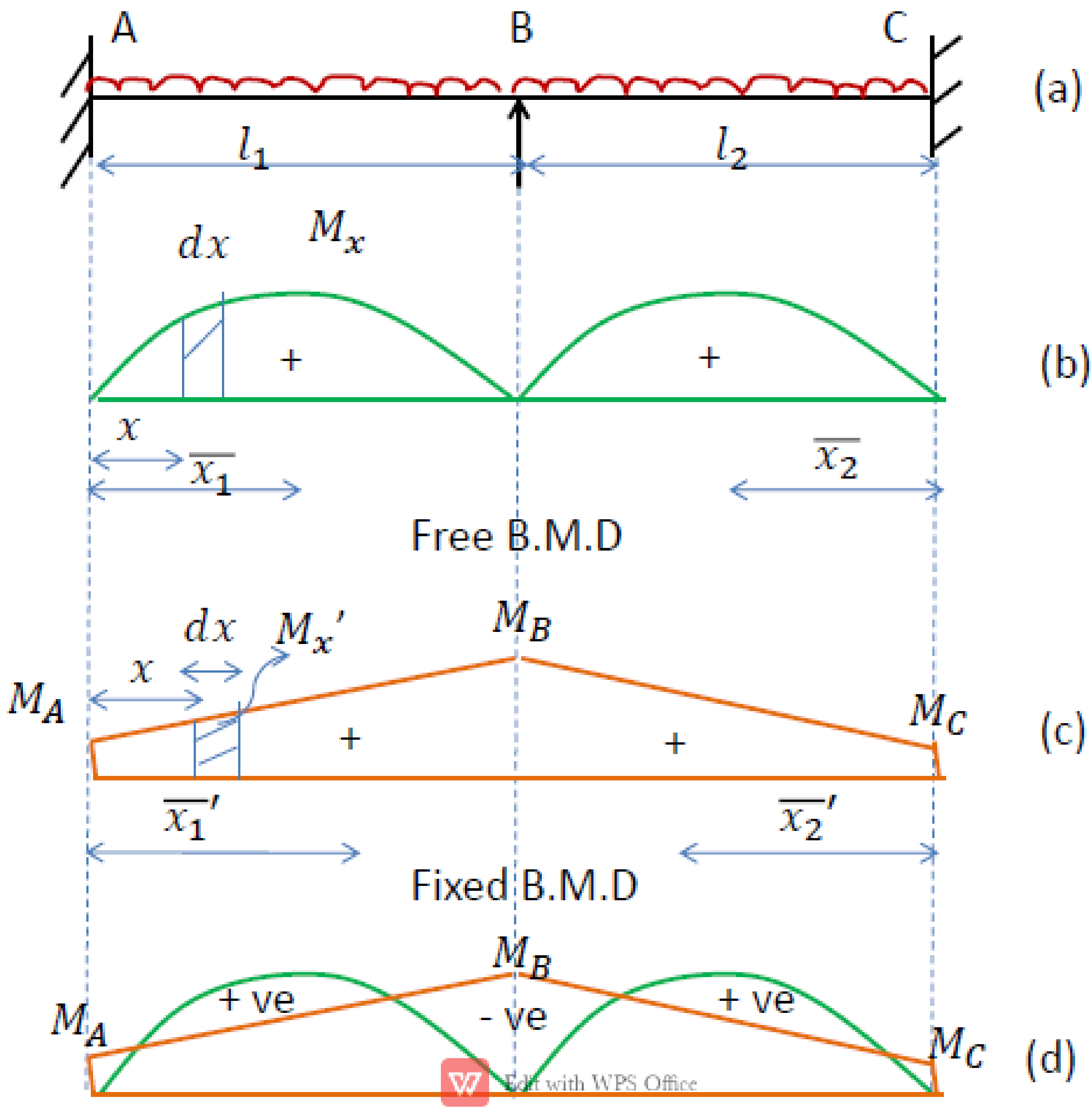
- As shown in above Figure, AB and BC are any two successive spans of a continuous beam subjected to an external loading.

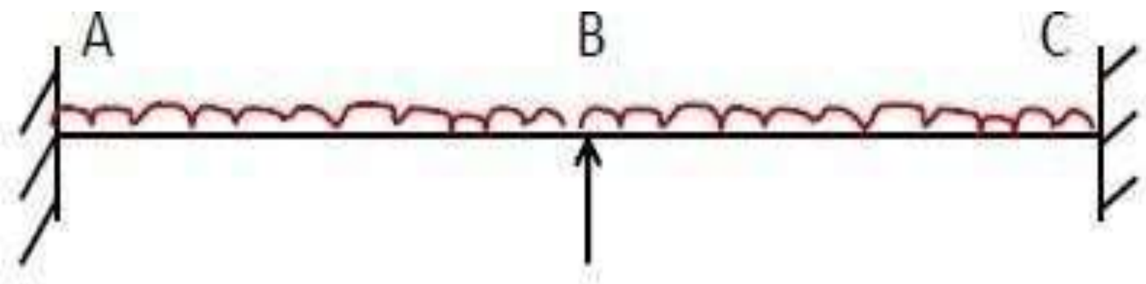
- If the extreme ends A and C fixed supports, the support moments M_A, M_B and M_C at the supports A, B and C are given by the relation,

$$M_A l_1 + 2M_B(l_1 + l_2) + M_C(l_2) = \frac{6a_1\overline{x_1}}{l_1} + \frac{6a_2\overline{x_2}}{l_2}$$

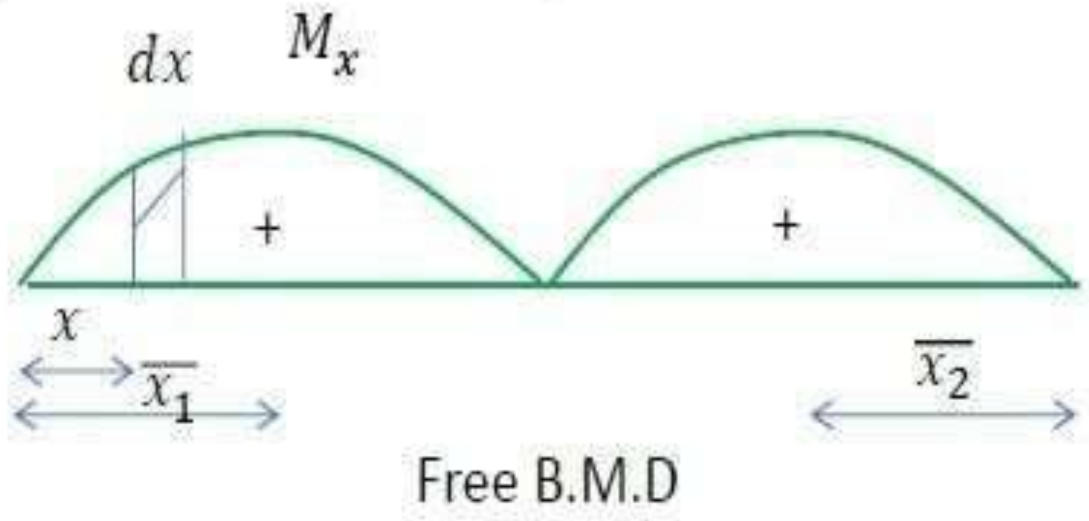
$$M_A l_1 + 2M_B(l_1 + l_2) + M_C(l_2) = \frac{6a_1\overline{x_1}}{l_1} + \frac{6a_2\overline{x_2}}{l_2}$$

- Where,
- a_1 = area of the free B.M. diagram for the span AB.
- a_2 = area of the free B.M. diagram for the span BC.
- $\overline{x_1}$= Centroidal distance of free B.M.D on AB from A.
- $\overline{x_2}$= Centroidal distance of free B.M.D on BC from C.

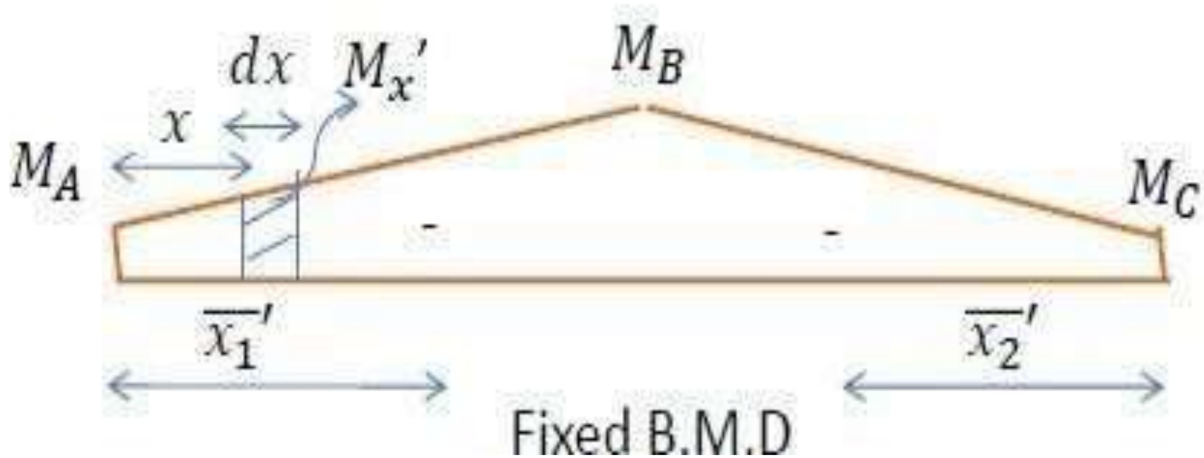

A
B
C
(a)
l_1
l_2
dx
M_x
+
+
(b)
x
$\overline{x_1}$
$\overline{x_2}$
Free B.M.D
dx
M_x'
M_B
x
M_A
M_C
+
+
(c)
$\overline{x_1}'$
$\overline{x_2}'$
Fixed B.M.D
M_B
+ ve
- ve
+ ve
M_A
M_C
(d)



(a)The given beam



(b) Free B.M.D.



(c) Fixed B.M.D.

- Consider the span AB:
- Let at any section in AB distant x from A the free and fixed bending moments be M_x and M_x' respectively.

- Hence the net bending moment at the section is given by

$$EI\frac{d^2y}{dx^2} = M_x - M_x'$$

- Multiplying by x, we get

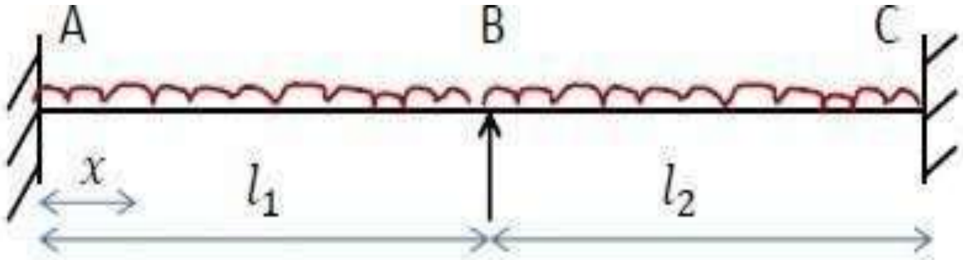
$$EIx\frac{d^2y}{dx^2} = M_x x - M_x' x$$

- $EIx\frac{d^2y}{dx^2} = M_x x - M_x' x$

- Integrating from $x = 0\ to\ x = l_1$, we get,

$$EI\int_0^{l_1} x\frac{d^2y}{dx^2} = \int_0^{l_1} M_x x\,dx - \int_0^{l_1} M_x' x\,dx$$

$$EI\left[x.\frac{dy}{dx} - y\right]_0^{l_1} = \int_0^{l_1} M_x x\,dx - \int_0^{l_1} M_x' x\,dx \qquad ---(1)$$



- But it may be such that

  At $x = 0$, deflection $y = 0$

- $At\ x = l_1, y = 0;\ and\ slope\ at\ B\ for AB, \frac{dy}{dx} = \theta_{BA}$

- $\int_0^{l_1} M_x x\,dx = a_1\overline{x_1}$ = Moment of the free B. M. D. on AB about A .

- $\int_0^{l_1} M_x{}'x\,dx = a_1'\ \overline{x_1}'$ = Moment of the fixed B. M. D. on AB about A.

$$EI\left[x.\frac{dy}{dx}-y\right]_0^{l_1} = \int_0^{l_1} M_x x\,dx - \int_0^{l_1} M_x{}' x\,dx \quad --(1)$$

- Therefore the equation (1) is simplified as,

$$EI\,[l_1\theta_{BA} - 0] = a_1\overline{x_1} - a_1'\overline{x_1}'.$$

But a_1' = area of the fixed B.M.D. on AB = $\frac{(M_A+M_B)}{2} l_1$

$\overline{x_1}'$ = Centroid of the fixed B. M. D. from A = $\frac{(M_A+2M_B)}{M_A+M_B}\frac{l_1}{3}$

- Therefore,

$$a_1'\overline{x_1}' = \frac{(M_A + M_B)}{2} l_1 \times \left(\frac{M_A + 2M_B}{M_A + M_B}\right) \frac{l_1}{3} = (M_A + 2M_B)\frac{l_1^2}{6}$$

$$EI\, l_1 \theta_{BA} = a_1\overline{x_1} - (M_A + 2M_B)\frac{l_1^2}{6}$$

$$6EI\,\theta_{BA} = \frac{6a_1\overline{x_1}}{l_1} - (M_A + 2M_B)l_1 \quad ----(2)$$

Similarly by considering the span BC and taking C as origin it can be shown that,

$$6EI\,\theta_{BC} = \frac{6a_2\overline{x_2}}{l_2} - (M_C + 2M_B)l_2 \quad ----(3)$$

θ_{BC} = slope for span CB at B

- But $\theta_{BA} = -\theta_{BC}$ as the direction of x from A for the span AB, and from C for the span CB are in opposite direction.
- And hence, $\theta_{BA} + \theta_{BC} = 0$

$$6EI\,\theta_{BA} = \frac{6a_1\overline{x_1}}{l_1} - (M_A + 2M_B)l_1 \quad ----(2)$$

$$6EI\,\theta_{BC} = \frac{6a_2\overline{x_2}}{l_2} - (M_C + 2M_B)l_2 \quad ----(3)$$

- Adding equations (2) and (3), we get

$$EI\,\theta_{BA} + 6EI\,\theta_{BC} = \frac{6a_1\overline{x_1}}{l_1} + \frac{6a_2\overline{x_2}}{l_2} - (M_A + 2M_B)l_1 - (M_C + 2M_B)l_2$$

$$6EI(\theta_{BA} + \theta_{BC}) = \frac{6a_1\overline{x_1}}{l_1} + \frac{6a_2\overline{x_2}}{l_2} - [M_A l_1 + 2M_B(l_1 + l_2) + M_C l_2]$$

$$0 = \frac{6a_1\overline{x_1}}{l_1} + \frac{6a_2\overline{x_2}}{l_2} - [M_A l_1 + 2M_B(l_1 + l_2) + M_C l_2]$$

$$[M_A l_1 + 2M_B(l_1 + l_2) + M_C l_2] = \frac{6a_1\overline{x_1}}{l_1} + \frac{6a_2\overline{x_2}}{l_2}$$

Problem 15.3. *A fixed beam AB of length 6 m carries point loads of 160 kN and 120 kN at a distance of 2 m and 4 m from the left end A. Find the fixed end moments and the reactions at the supports. Draw B.M. and S.F. diagrams.*

Sol. Given :

Length = 6 m

Load at C, $W_C = 160$ kN

Load at D, $W_D = 120$ kN

Distance $AC = 2$ m

Distance $AD = 4$ m

For the sake of convenience, let us first calculate the fixed end moments due to loads at C and D and then add up the moments.

(*i*) Fixed end moments due to load at C.

For the load at C, $a = 2$ m and $b = 4$ m

$$\therefore \quad M_{A_1} = \frac{W_C . a . b^2}{L^2}$$

$$= \frac{160 \times 2 \times 4^2}{6^2} = 142.22 \text{ kNm}$$

$$M_{B_1} = \frac{W_C . a^2 . b}{L^2} = \frac{160 \times 2^2 \times 4}{6^2} = 71.11 \text{ kNm}$$

(*ii*) Fixed end moments due to load at D.

Similarly for the load at D, $a = 4$ m and $b = 2$ m

$$\therefore \quad M_{A_2} = \frac{W_D . a . b^2}{L^2}$$

$$= \frac{120 \times 4 \times 2^2}{6^2} = 53.33 \text{ kNm}$$

and $$M_{B_2} = \frac{W_D . a^2 . b}{L^2} = \frac{160 \times 4^2 \times 2}{6^2} = 106.66 \text{ kNm}$$

Total fixing moment at A,

$$M_A = M_{A_1} + M_{A_2} = 142.22 + 53.33$$

$$= \mathbf{195.55 \text{ kNm.}} \quad \textbf{Ans.}$$

and total fixing moment at B,

$$M_B = M_{B_1} + M_{B_2} = 71.11 + 106.66$$

$$= \mathbf{177.77 \text{ kNm.}} \quad \textbf{Ans.}$$

B.M. diagram due to vertical loads

Consider the beam AB as simply supported. Let $R_A{}^*$ and $R_B{}^*$ are the reactions at A and B due to simply supported beam. Taking moments about A, we get

$$R_B{}^* \times 6 = 160 \times 2 + 120 \times 4$$

$$= 320 + 480 = 800$$

$$\therefore \quad R_B{}^* = \frac{800}{6} = 133.33 \text{ kN}$$

and $$R_A{}^* = \text{Total load} - R_B{}^* = (160 + 120) - 133.33$$

$$= 146.67 \text{ kN}$$

B.M. at $A = 0$

B.M. at $C = R_A{}^* \times 2 = 146.67 \times 2 = 293.34$ kNm

B.M. at $D = R_B{}^* \times 2 = 133.33 \times 2 = 266.66$ kNm

B.M. at $B = 0$.

S.F. Diagram

Let R_A = Resultant reaction at A due to fixed end moments and vertical loads

R_B = Resultant reaction at B.

Equating the clockwise moments and anti-clockwise moments about A, we get

$$R_B \times 6 + M_A = 160 \times 2 + 120 \times 4 + M_B$$

$$R_B \times 6 + 195.55 = 320 + 480 + 177.77$$

$$\therefore \quad R_B = \frac{800 + 177.77 - 195.55}{6} = 130.37 \text{ kN}$$

$$R_A = \text{Total load} - R_B$$

$$= (160 + 120) - 130.37 = 149.63 \text{ kN}$$

S.F. at $A = R_A = 149.63$ kN

S.F. at $C = 149.63 - 160 = -10.37$ kN

S.F. at $D = -10.37 - 120 = -130.37$ kN

S.F. at $B = -130.37$ kN

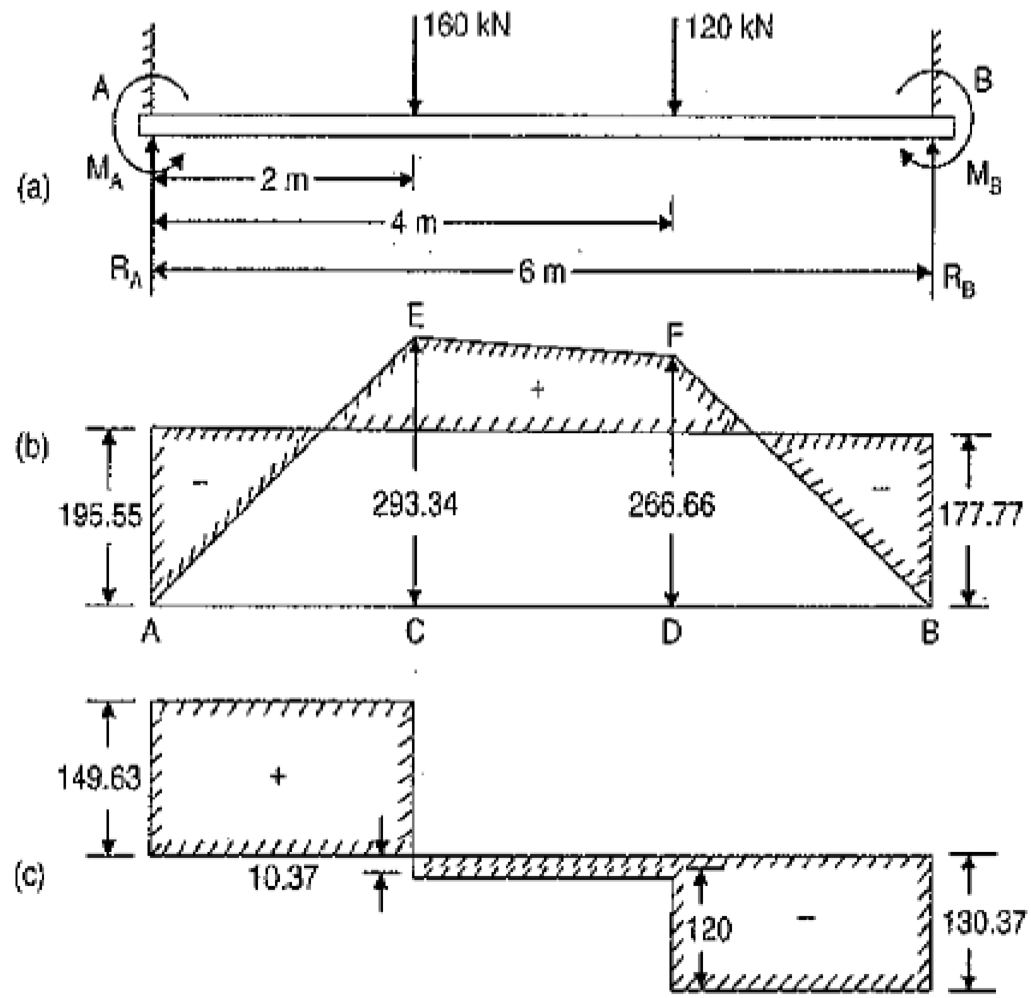


Fig. 15.8

Problem 15.6. *Find the fixing moments and support reactions of a fixed beam AB of length 6 m, carrying a uniformly distributed load of 4 kN/m over the left half of the span.*

Macaulay's method can be used and directly the fixing moments and end reactions can be calculated. This method is used where the areas of B.M. diagrams cannot be determined conveniently.

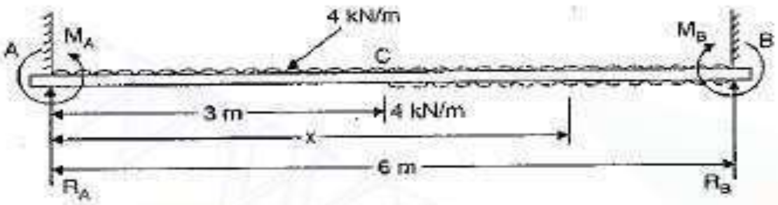


Fig. 15.12

For this method it is necessary that u.d.l. should be extended upto B and then compensated for upward u.d.l. for length BC as shown in Fig. 15.12.

The B.M. at any section at a distance x from A is given by

$$EI\frac{d^2y}{dx^2} = R_A.x - M_A - w \times x \times \frac{x}{2} \quad \Big| \quad + w \times (x-3) \times \frac{(x-3)}{2}$$

$$= R_A x - M_A - \frac{4 \times x^2}{2} \quad \Big| \quad + \frac{4(x-3)^2}{2}$$

$$= R_A.x - M_A - 2x^2 \quad \Big| \quad + 2(x-3)^2 \qquad ...(A)$$

Integrating, we get

$$EI\frac{dy}{dx} = R_A.\frac{x^2}{2} - M_A.x - \frac{2x^3}{3} + C_1 \quad \Big| \quad + \frac{2(x-3)^3}{3} \qquad ...(i)$$

when $x = 0$, $\frac{dy}{dx} = 0$.

Substituting this value in the above equation upto dotted line, we get

$C_1 = 0.$

Therefore equation (*i*) becomes as

$$EI\frac{dy}{dx} = R_A.\frac{x^2}{2} - M_A.x - \frac{2x^3}{3} \quad \Big| \quad + \frac{2(x-3)^3}{3} \qquad ...(ii)$$

Integrating again, we get

$$EI\,y = \frac{R_A}{2}.\frac{x^3}{3} - \frac{M_A.x^2}{2} - \frac{2}{3}\frac{x^4}{4} + C_2 \quad \Big| \quad + \frac{2}{3}\frac{(x-3)^4}{4} \qquad ...(iii)$$

when $x = 0$, $y = 0$.

Substituting this value upto dotted line, we get

$C_2 = 0$

Therefore equation (*iii*) becomes as

$$EI\,y = \frac{R_A.x^3}{6} - \frac{M_A.x^2}{2} - \frac{1}{6}x^4 \quad \Big| \quad + \frac{1}{6}(x-3)^4 \qquad ...(iv)$$

when $x = 6$, $y = 0$.

Substituting this value in equation (*iv*) [Here complete equation is taken], we get

$$0 = \frac{R_A \times 6^3}{6} - \frac{M_A \times 6^2}{2} - \frac{1}{6} \times 6^4 + \frac{1}{6} \times (6-3)^4$$

$$= 36R_A - 18M_A - 216 + 13.5$$

$$202.50 = 36R_A - 18M_A$$

$$101.25 = 18R_A - 9M_A \qquad ...(v)$$

At $x = 6$ m, $\frac{dy}{dx} = 0$.

Substituting these values in the complete equation (*ii*), we get

$$0 = R_A \times \frac{6^2}{2} - M_A \times 6 - \frac{2}{3} \times 6^3 + \frac{2}{3}(6-3)^3$$

$$= 18R_A - M_A \times 6 - 144 + 18$$

$$126 = 18R_A - 6M_A \quad \text{...}(\overline{vi})$$

Substracting equation (*v*) from equation (*vi*), we get

$$126 - 101.25 = 9M_A - 6M_A$$

or $$24.75 = 3M_A$$

$\therefore$ $$M_A = \frac{24.75}{3} = \mathbf{8.25\ kNm.}\quad \textbf{Ans.}$$

Substituting this value in equation (*vi*), we get

$$126 = 18R_A - 6 \times 8.25$$

$\therefore$ $$R_A = \frac{126 + 6 \times 8.25}{18} = \mathbf{9.75\ kN.}\quad \textbf{Ans.}$$

Now $$R_B = \text{Total load} - R_A$$

$$= 4 \times 3 - 9.75 = \mathbf{2.25\ kN.}\quad \textbf{Ans.}$$

To find the value of M_B, we must equate the clockwise moments and anti-clockwise moments about *B*. Hence

Clockwise moments about B = Anti-clockwise moments about B.

$$M_B + R_A \times 6 = M_A + 4 \times 3 \times (4.5)$$

or $$M_B + 9.75 \times 6 = 8.25 + 54 \quad (\because R_A = 9.75 \text{ and } M_A = 8.25)$$

or $$M_B + 58.50 = 62.25$$

$\therefore$ $$M_B = 62.25 - 58.50 = \mathbf{3.75\ kNm.}\quad \textbf{Ans.}$$

Problem 15.7. *A fixed beam of length 20 m, carries a uniformly distributed load of 8 kN/m on the left hand half together with a 120 kN load at 15 m from the left hand end. Find the end reactions and fixing moments and magnitude and the position of the maximum deflection. Take $E = 2 \times 10^8$ kN/m^2 and $I = 4 \times 10^8$ mm^4.*

Sol. Given :

Length, $L = 20$ m

U.d.l., $w = 8$ kN/m

Point load, $W = 120$ kN

Value of $E = 2 \times 10^8$ kN/m^2

Value of $I = 4 \times 10^8 \text{ mm}^4 = 4 \times 10^{-4} \text{ m}^4$

Lengths, $AC = 10$ m, $AD = 15$ m

Fig. 15.13 shows the loading on the fixed beam.

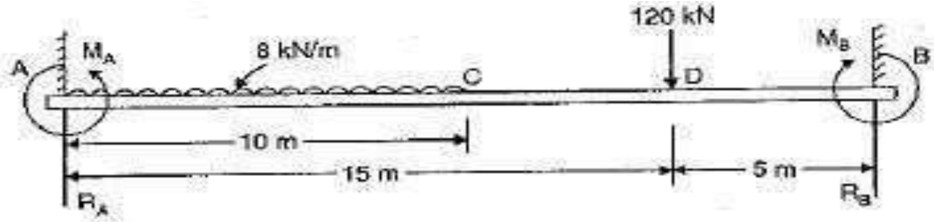


Fig. 15.13

Let R_A and R_B = End reactions at A and B

M_A and M_B = Fixing moments at A and B

Let us apply Macaulay's method for this case. Hence it is necessary that the u.d.l. should be extended upto B and then compensated for upward u.d.l. for length BC as shown in Fig. 15.14.

The B.M. at any section at a distance x from A is given by,

$$EI\frac{d^2y}{dx^2} = R_A.x - M_A - w \times x \times \left(\frac{x}{2}\right) \;\vdots\; - 120(x - 15) \;\vdots\; + w \times (x - 10) \times \left(\frac{x - 10}{2}\right)$$

$$= R_A \times x - M_A - 8 \times \frac{x^2}{2} \;\vdots\; - 120(x - 15) \;\vdots\; + \frac{8 \times (x - 10)^2}{2}$$

$$= R_A.x - M_A - 4x^2 \;\vdots\; - 120(x - 15) \;\vdots\; + 4(x - 10)^2$$

Integrating the above equation, we get

$$EI\frac{dy}{dx} = R_A.\frac{x^2}{2} - M_A.x - 4.\frac{x^3}{3} + C_1 \;\vdots\; - \frac{120(x - 15)^2}{2} \;\vdots\; + \frac{4(x - 10)^3}{3} \quad ...(i)$$

when $x = 0$, $\frac{dy}{dx} = 0$. Substituting this value in the above equation upto first dotted line, we get $C_1 = 0$. Therefore, equation (i) becomes as

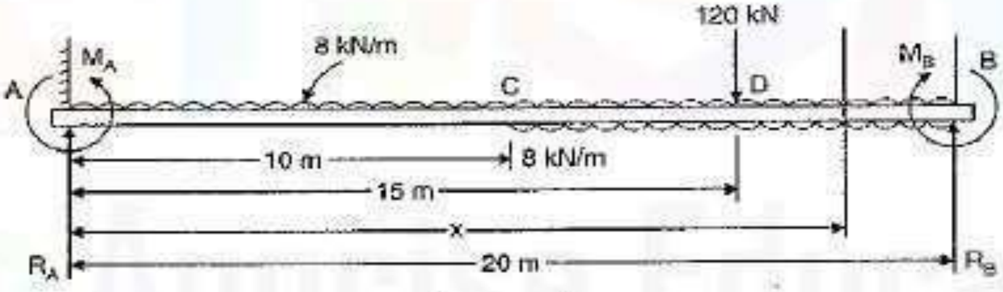


Fig. 15.14

$$EI\frac{dy}{dx} = \frac{R_A}{2}.x^2 - M_A.x - \frac{4}{3}x^3 \;\vdots\; - 60(x - 15)^2 \;\vdots\; + \frac{4}{3}(x - 10)^3 \quad ...(ii)$$

Integrating again, we get

$$EIy = \frac{R_A . x^3}{6} - \frac{M_A . x^2}{2} - \frac{4x^4}{3 \times 4} + C_2 \; \vdots - \frac{60(x-15)^3}{3} \; \vdots + \frac{4}{3}\frac{(x-10)^4}{4} \quad ...(iii)$$

when $x = 0$, $y = 0$. Substituting this value in the above equation upto first dotted line, we get $C_2 = 0$. Therefore equation (*iii*) becomes as

$$EIy = \frac{R_A . x^3}{6} - \frac{M_A . x^2}{2} - \frac{x^4}{3} \; \vdots - 20(x-15)^3 \; \vdots + \frac{1}{3}(x-10)^4 \quad ...(iv)$$

when $x = 20$, $y = 0$. Substituting these values in complete equation (*iv*), we get

$$0 = \frac{R_A \times 20^3}{6} - \frac{M_A \times 20^2}{2} - \frac{20^4}{3} - 20(20-15)^3 + \frac{1}{3}(20-10)^4$$

$$= \frac{20}{6} R_A - \frac{M_A}{2} - \frac{20^2}{3} - \frac{125}{20} + \frac{1}{3} \times \frac{10^4}{400} \quad \text{(Dividing by } 20^2\text{)}$$

$$= \frac{20}{6} R_A - \frac{M_A}{2} - \frac{400}{3} - \frac{12.5}{2} + \frac{25}{3}$$

$$= \frac{20R_A - 3M_A - 800 - 37.5 + 50}{6}$$

or $\quad 20R_A - 3M_A = 800 + 37.5 - 50 = 787.5 \quad ...(v)$

At $x = 20$, $\frac{dy}{dx} = 0$. Substituting these values in complete equation (*ii*), we get

$$0 = \frac{R_A}{2} \times 20^2 - M_A \times 20 - \frac{4}{3} \times 20^3 - 60(20-15)^2 + \frac{4}{3}(20-10)^3$$

$$= \frac{20}{6} R_A - \frac{M_A}{2} - \frac{20^2}{3} - \frac{125}{20} + \frac{1}{3} \times \frac{10^4}{400} \quad \text{(Dividing by } 20^2\text{)}$$

$$= \frac{20}{6} R_A - \frac{M_A}{2} - \frac{400}{3} - \frac{12.5}{2} + \frac{25}{3}$$

$$= \frac{20R_A - 3M_A - 800 - 37.5 + 50}{6}$$

or $\quad 20R_A - 3M_A = 800 + 37.5 - 50 = 787.5 \quad ...(v)$

At $x = 20$, $\frac{dy}{dx} = 0$. Substituting these values in complete equation (*ii*), we get

$$0 = \frac{R_A}{2} \times 20^2 - M_A \times 20 - \frac{4}{3} \times 20^3 - 60(20-15)^2 + \frac{4}{3}(20-10)^3$$

$$= 10R_A - M_A - \frac{4 \times 400}{3} - 3 \times 25 + \frac{4}{3} \times \frac{1000}{20} \quad \text{(Dividing by 20)}$$

$$= 10R_A - M_A - \frac{1600}{3} - 75 + \frac{200}{3}$$

or $\quad 10R_A - M_A = \frac{1600}{3} + 75 - \frac{200}{3} = \frac{1400}{3} + 75$

or $\quad 10R_A - M_A = 541.66$

or $\quad 20R_A - 2M_A = 1083.32 \quad$ (Multiplying by 2 both sides) $\quad ...(vi)$

Substracting equation (*v*) from equation (*vi*), we get

$$M_A = 1083.32 - 787.50 = \textbf{295.82 kNm.} \quad \textbf{Ans.}$$

Substituting this values of M_A in equation (*vi*), we get

$$20R_A - 2 \times 295.82 = 1083.32$$

$$\therefore \quad R_A = \frac{1083.32 + 2 \times 295.82}{20}$$

$$= \textbf{83.748 kN.} \quad \textbf{Ans.}$$

Now $\quad R_B = \text{Total load on beam} - R_A$

$$= (10 \times 8 + 120) - 83.748 = \textbf{116.252 kN.} \quad \textbf{Ans.}$$

Equating the clockwise moment and anti-clockwise moment about B, we get

$$M_B + R_A \times 20 = M_A + 120 \times 5 + 8 \times 10 \times 15$$

or $\quad M_B + 83.748 \times 20 = 295.82 + 600 + 1200$

or $\quad M_B = 2095.82 - 83.748 \times 20 = \textbf{420.86 kNm.} \quad \textbf{Ans.}$

Problem 15.8.s A *fixed beam AB of length 3 m is having moment of inertia $I = 3 \times 10^6$ mm^4. The support B sinks down by 3 mm. If $E = 2 \times 10^5$ N/mm^2, find the fixing moments.*

Sol. Given :

Length, $L = 3 \text{ m} = 3000 \text{ mm}$

Value of $I = 3 \times 10^6 \text{ mm}^4$

Value of $E = 2 \times 10^5 \text{ N/mm}^2$

The amount by which the support B sinks down,

$$\delta = 3 \text{ mm.}$$

The fixing moments at the ends is given by,

$$M_A = M_B = \frac{6EI\delta}{L^2}$$

$$= \frac{6 \times 2 \times 10^5 \times 3 \times 10^6 \times 3}{3000^2}$$

$$= 12 \times 10^5 \text{ Nmm} = 12 \times 10^3 \text{ Nm} = \mathbf{12 \text{ kNm.}} \quad \textbf{Ans.}$$

The fixing moment at A will be a hogging moment whereas at B it will be a sagging moment.

EXAMPLE 24.5. *A fixed beam AB of span 6 m is carrying a uniformly distributed load of 4kN/m over the left half of the span. Find the fixing moments and support reactions.*

SOLUTION. Given: Span (l) = 6 m ; Uniformly distributed load (w) = 4 kN/m and loaded portion (l_1) = 3 m.

Fixing moments

Let M_A = Fixing moment at A and,

M_B = Fixing moment at B.

First of all, consider the beam AB on a simply supported. Taking moments about A,

$$R_B \times 6 = 4 \times 3 \times 1.5 = 18$$

$$\therefore \quad R_B = \frac{18}{6} = 3 \text{ kN}$$

and $$R_A = 3 \times 4 - 3 = 9 \text{ kN}$$

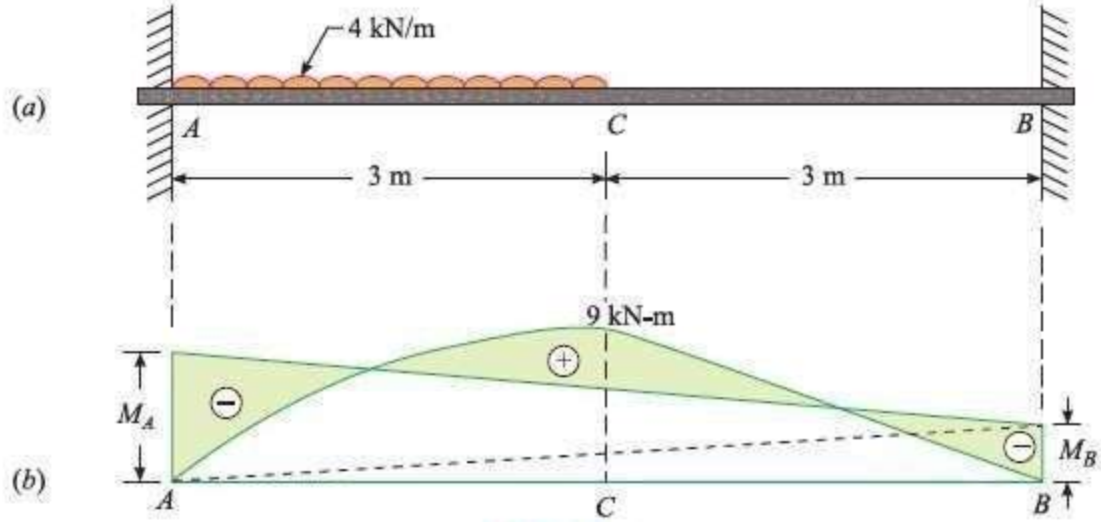


Fig. 24.6

We know that μ-diagram will be parabolic from A to C and triangular from C to B as shown in ig. 24.6 (b). The bending moment at C (treating the beam as a simply supported),

$$M_C = R_B \times 3 = 3 \times 3 = 9 \text{ kN-m}$$

The bending moment at any section X in AC, at a distance x from A (treating the beam as a simply ıpported),

$$M_X = 9x - 4x \cdot \frac{x}{2} = 9x - 2x^2$$

∴ Area μ-diagram from A to B,

$$a = \int_0^3 (9x - 2x^2)\,dx + \frac{1}{2} \times 9.0 \times 3$$

$$= \left[\frac{9x^2}{2} - \frac{2x^3}{3}\right]_0^3 + 13.5$$

$$= \frac{9 \times (3)^2}{2} - \frac{2 \times (3)^3}{3} + 13.5 = 36$$

nd area of μ′-diagram, $$a' = (M_A + M_B) \times \frac{6}{2} = 3\,(M_A + M_B)$$

We know that $$a' = -a$$

$$\therefore \quad 3\,(M_A + M_B) = -36$$

or $$M_A + M_B = -\frac{36}{3} = -12$$

Moment of μ-diagram area about A (by splitting up the diagram into AC and CB),

$$a\bar{x} = \int_0^3 (9x^2 - 2x^3)\,dx + \frac{1}{2}\times 9\times 3\times 4$$

$$a\bar{x} = \left[\frac{9x^3}{3} - \frac{2x^4}{4}\right]_0^3 + 54$$

$$= \left[\frac{9\times(3)^3}{3} - \frac{2\times(3)^4}{4}\right] + 54 = 94.5$$

and moment of $\bar{x}'$-diagram area about A (by splitting up the trapezium into two triangles) as shown in Fig. 24.6 (a),

$$a'\bar{x}' = \left(M_A\times\frac{6}{2}\times\frac{6}{3}\right) + M_B\times\frac{6}{2}\times\frac{2\times 6}{3}$$

$$= 6M_A + 12M_B = 6(M_A + 2M_B)$$

We know that $a'\bar{x} = -a\bar{x}$

$$6(M_A + 2M_B) = -94.5$$

$$\therefore \quad M_A + 2M_B = -\frac{94.5}{6} = -15.75 \qquad ...(ii)$$

Solving equations (i) and (ii),

M_A = **– 8.25 kN-m** **Ans.**

M_B = **– 3.75 kN-m** **Ans.**

Now complete the bending moment diagram as shown in Fig. 24.6 (b).

Support reactions

Let R_A = Reaction at A, and

R_B = Reaction at B.

Equating the clockwise moments and anticlockwise moments about A,

$$R_B\times 6 + 8.25 = (4\times 3\times 1.5) + 3.75 = 21.75$$

$\therefore \quad R_B = \dfrac{21.75 - 8.25}{6}$ = **2.25 kN** **Ans.**

and $R_A = 4\times 3 - 2.25$ = **9.75 kN** **Ans.**

EXAMPLE 24.6. *A beam AB of uniform section and 6 m span is built-in at the ends. A uniformly distributed load of 3 kN/m runs over the left half of the span and there is in addition a concentrated load of 4 kN at right quarter as shown in Fig. 24.7.*

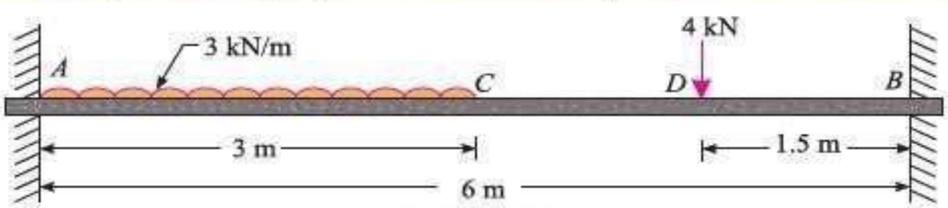


Fig. 24.7

Determine the fixing moments at the ends, and the reactions. Sketch neatly the bending moment and shearing force diagram marking thereon salient values.

SOLUTION: Given: Span (l) = 6 m ; Uniformly distributed load on AC (w) = 3 kN/m ; Loaded portion (l_1) = 3 m and concentrated load at D (W) = 4 kN.

Fixing moments at the ends

Let M_A = Fixing moment at A and

First of all, consider the beam AB as a simply supported. Taking moments about A,

$$R_B \times 6 = (3 \times 3 \times 1.5) + (4 \times 4.5) = 31.5$$

$$\therefore \quad R_B = \frac{31.5}{6} = 5.25 \text{ kN}$$

and

$$R_A = (3 \times 3 + 4) - 5.25 = 7.75 \text{ kN}$$

We know that the μ-diagram will be parabolic from A to C, trapezoidal from C to D and triangular from D to B as shown in Fig. 24.8(b). The bending moment at D (treating the beam at a simply supported),

$$M_D = 5.25 \times 1.5 = 7.875 \text{ kN-m}$$

and

$$M_C = 5.25 \times 3 - 4 \times 1.5 = 9.75 \text{ kN-m}$$

The bending moment at any section X in AC, at a distance x from A (treating the beam as a simply supported),

$$M_X = 7.75x - 3x\,\frac{x}{2} = 7.75x - 1.5x^2$$

∴ Area of μ-diagram from A to B,

$$\therefore \quad a = \int_0^3 (7.75x - 1.5x^2)\,dx + \left(\frac{1}{2}(9.75 + 7.875) \times 1.5\right) + \left(\frac{1}{2} \times 7.875 \times 1.5\right)$$

$$= \left[\frac{7.75x^2}{2} - \frac{1.5x^3}{3}\right]_0^3 + 19.125$$

$$= \frac{7.75 \times (3)^2}{2} - \frac{1.5 \times (3)^3}{3} + 19.125 = 40.5$$

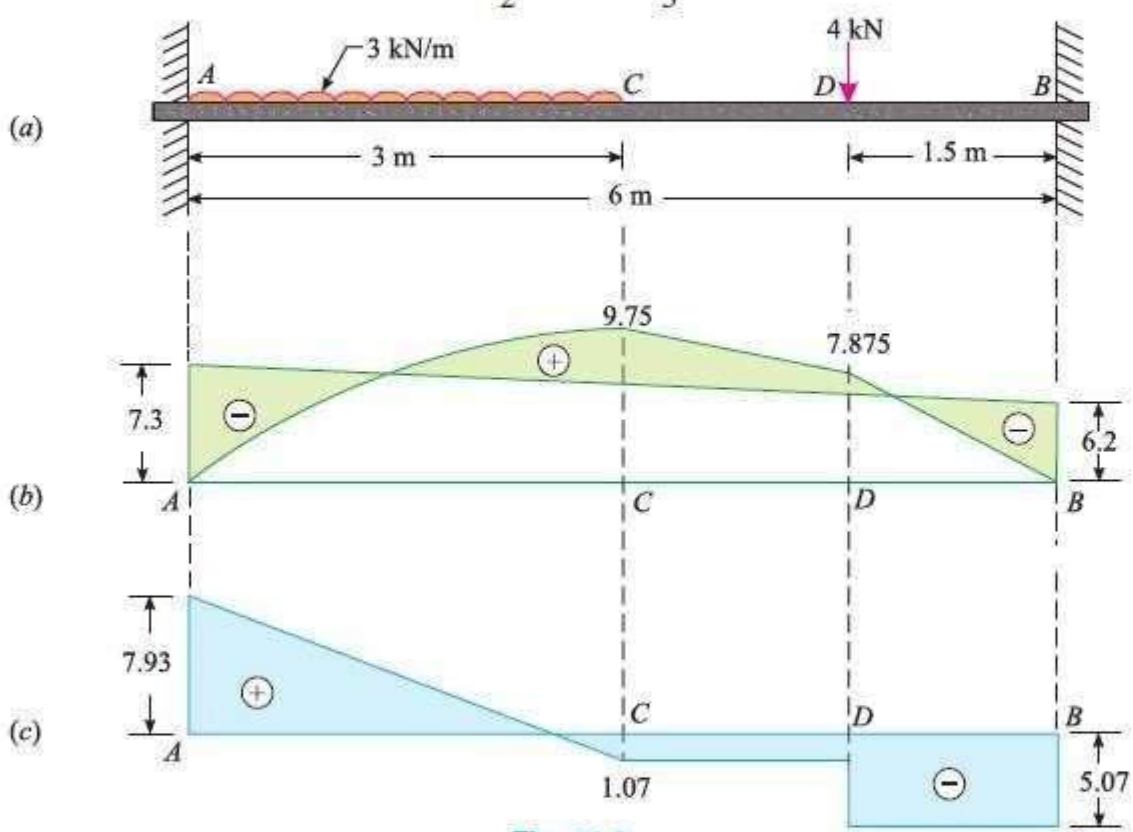


Fig. 24.8

and area of μ-diagram, $a' = (M_A + M_B) \times \frac{6}{2} = 3\,(M_A + M_B)$

We know that $a' = -a$

$$\therefore \quad 3\,(M_A + M_B) = -40.5 \qquad ...(\because a = 40.5)$$

$$\text{or} \quad M_A + M_B = -13.5 \qquad ...(i)$$

Moment of μ-diagram area about *A* (by splitting up the diagram into *AC*, *CD* and *DB*),

$$a\bar{x} = \int_0^3 (7.75x^2 - 1.5x^3)\,dx + \left(\frac{1}{2} \times 9.75 \times 1.5 \times 3.5\right)$$

$$+ \left(\frac{1}{2} \times 7.875 \times 1.5 \times 4\right) + \left(\frac{1}{2} \times 7.875 \times 1.5 \times 5\right)$$

$$= \left[\frac{7.75x^3}{3} - \frac{1.5x^4}{4}\right]_0^3 + 78.75$$

$$= \left[\frac{7.75 \times (3)^3}{3} - \frac{1.5 \times (3)^4}{4}\right] + 78.75 = 118.1$$

and moment of μ′-diagram area about *A* (by splitting up the trapezium into two triangles),

$$a'\bar{x}' = \left(M_A \times \frac{6}{2} \times \frac{6}{3}\right) + \left(M_B \times \frac{6}{2} \times \frac{2 \times 6}{3}\right)$$

$$= 6M_A + 12M_B = 6\,(M_A + 2M_B)$$

We know that $a'\bar{x}' = -a\bar{x}$

$$\therefore \quad 6\,(M_A + 2M_B) = -118.1$$

$$\text{or} \quad M_A + 2M_B = -\frac{118.1}{6} = -19.7 \qquad ...(ii)$$

Solving equations (*i*) and (*ii*), we get

$$M_A = -7.3 \text{ kN-m} \quad \text{and} \quad M_B = -6.2 \text{ kN-m}$$

Now complete the bending moment diagram as shown in Fig. 24.8 (*b*).

Shearing force diagram

Let R_A = Reaction at *A* and

R_B = Reaction at *B*.

Equating the clockwise moments and anticlockwise moments about *A*,

$$R_B \times 6 + 7.3 = (3 \times 3 \times 1.5) + (4 \times 4.5) + 6.2 = 37.7$$

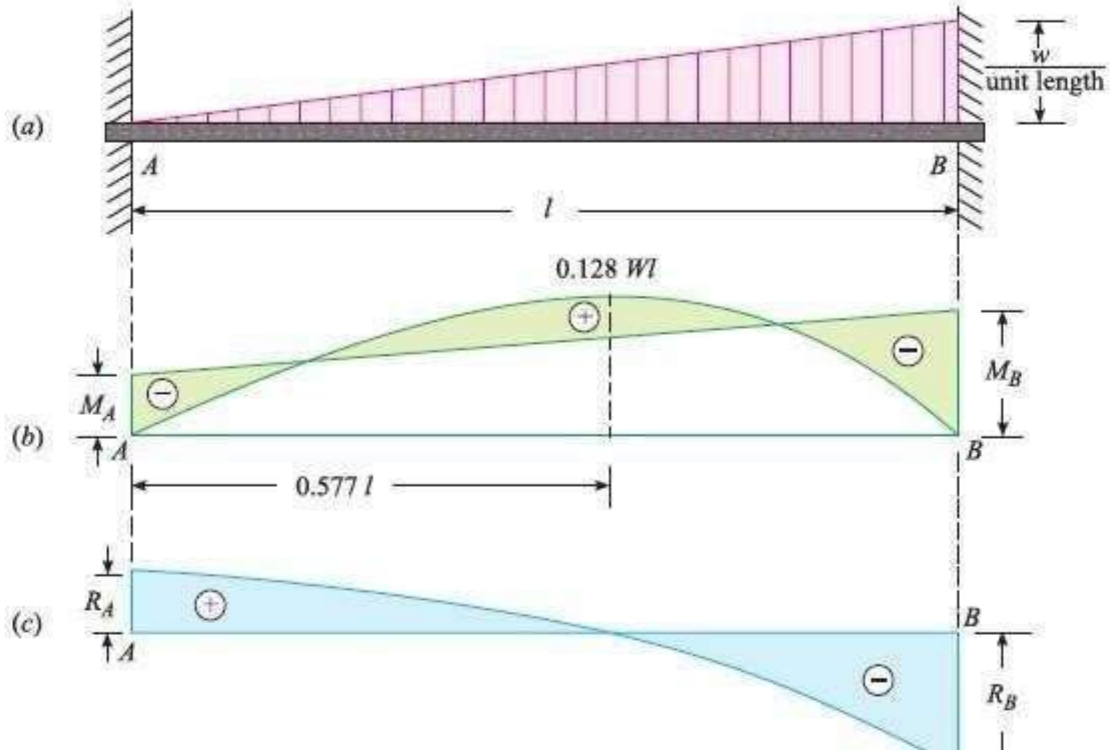
$$\therefore \quad R_B = \frac{37.7 - 7.3}{6} = 5.07 \text{ kN}$$

$$\text{and} \quad R_A = (3 \times 3 + 4) - 5.07 = 7.93 \text{ kN}$$

Now complete the shear force diagram as shown in Fig. 24.8 (*c*).

24.8. Fixing Moments of a Fixed Beam Carrying a Gradually Varying Load from Zero at One End to *w* per unit length at the Other

Consider a beam *AB* fixed at *A* and *B* and carrying a gradually varying load from zero at *A* to *w* per unit length at *B* as shown in Fig. 24.9 (*a*).



Let l = Span of the beam,

M_A = Fixing moment at *A* and

M_B = Fixing moment at *B*.

First of all, consider the beam *AB* as a simply supported and taking moments about *A*,

$$R_B \times l = w \times \frac{l}{2} \times \frac{2l}{3} = \frac{wl^2}{3}$$

$$\therefore \quad R_B = \frac{wl}{3}$$

and

$$R_A = \frac{wl}{2} - \frac{wl}{3} = \frac{wl}{6}$$

We know that the μ-diagram will be parabolic from *A* to *B*. The bending moment at any section *X*, at a distance *x* from *A* (treating the beam as a simply supported),

$$M_X = \frac{wl}{6} \times x - \frac{wx}{l} \times \frac{x}{2} \times \frac{x}{3} = \frac{wlx}{6} - \frac{wx^3}{6l}$$

∴ Area of μ-diagram,

$$a = \int_0^l \left(\frac{wlx}{6} - \frac{wx^3}{6l} \right) dx$$

$$= \frac{w}{6} \int_0^l \left(lx - \frac{x^3}{l} \right) dx$$

$$= \frac{w}{6} \left[\frac{lx^2}{2} - \frac{x^4}{4l} \right]_0^l$$

$$= \frac{w}{6} \left(\frac{l^3}{2} - \frac{l^3}{4} \right) = \frac{wl^3}{24}$$

and area of μ′-diagram, $a' = \frac{l}{2}(M_A + M_B)$

We know that $a' = -a$

$$\therefore \quad \frac{l}{2}(M_A + M_B) = -\frac{wl^3}{24}$$

or $$M_A + M_B = -\frac{wl^2}{12}$$

Moment of μ-diagram area about A,

$$a\bar{x} = \int_0^l \left(\frac{wlx^2}{6} - \frac{wx^4}{6l}\right) dx$$

$$= \frac{w}{6}\int_0^l \left(lx^2 - \frac{x^4}{l}\right) dx$$

$$= \frac{w}{6}\left[\frac{lx^3}{3} - \frac{x^5}{5l}\right]_0^l$$

$$= \frac{w}{6}\left(\frac{l^4}{3} - \frac{l^4}{5}\right) = \frac{wl^4}{45}$$

and moment of μ′-diagram about A (by splitting up the trapezium into two triangles),

$$a' x' = M_A \times \frac{l}{2} \times \frac{l}{3} + M_B \times \frac{l}{2} \times \frac{l}{3}$$

$$= \frac{l^2}{6}(M_A + 2M_B)$$

We know that $a'\bar{x}' = -a\bar{x}$

$$\therefore \quad \frac{l^2}{6}(M_A + 2M_B) = -\frac{wl^4}{45}$$

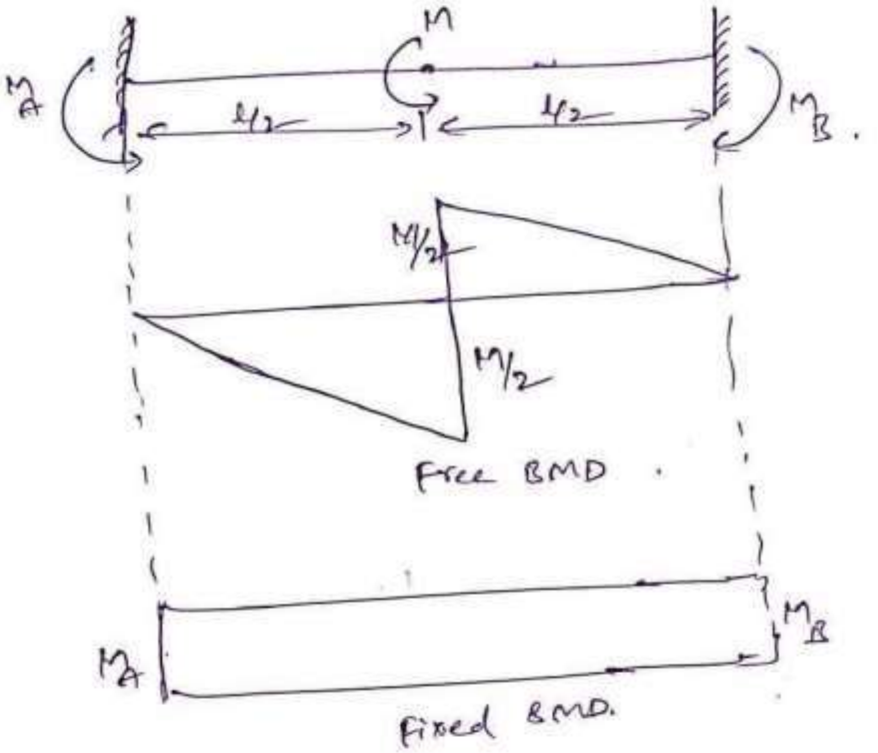
or $$M_A + 2M_B = -\frac{2wl^2}{15} \quad \text{...}(ii)$$

Solving equations (*i*) and (*ii*),

$$M_A = -\frac{wl^2}{30} = -\frac{Wl}{15} \quad \text{...}\left(\because W = \frac{wl}{2}\right)$$

and $$M_B = -\frac{wl^2}{20} = -\frac{Wl}{10} \quad \text{...}\left(\because W = \frac{wl}{2}\right)$$

Fixed Beam Subjected to Concentrated Couple at mid span :-



Area of free BMD = $A = \left(\frac{1}{2} \times \frac{l}{2} \times \frac{M}{2}\right) \times 2 = \frac{Ml}{4}$

Area of fixed BMD = $A' = M_A \times l$

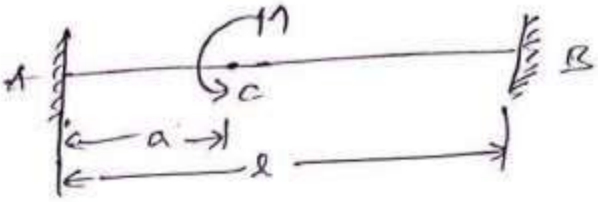
$$A' = A$$

$$M_A l = \frac{Ml}{4}$$

$$\boxed{M_A = M_B = \frac{M}{4}}$$

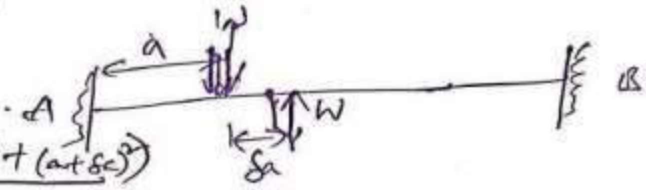
Fixed Beam subjected to a couple applied eccentrically on the span-

$M_A = \frac{Wab^2}{l^2}$



$W = W\uparrow + W\downarrow$

$\therefore M_A = \frac{W(a)(l-a)^2}{l^2} - \frac{W(a+\delta a)(l-(a+\delta a))^2}{l^2}$



$= \frac{Wa(l-a)^2 - W(a+\delta a)(l^2 - 2l(a+\delta a) + (a+\delta a)^2)}{l^2}$

as δa is small, $(\delta a)^2$ very small, so neglected.

$\therefore M_A = \frac{Wa(l-a)^2 - W[al^2 + 2al(a+\delta a) + \delta a l^2]}{l^2}$

$= \frac{Wa(l-a)^2 - W(a+\delta a)[(l-a)^2 - 2\delta a(l-a) + (\delta a)^2]}{l^2}$

$= \frac{W\delta a(la)}{l^2}[2a-(l-a)]$

$= \frac{-W\delta a}{l^2}[(l-a)(l-3a)]$

If δa is small Moment $M = W \times \delta a$.

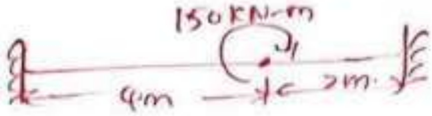
$\therefore M_A = \frac{-M(l-a)(l-3a)}{l^2}$

$M_B = \frac{Wa^2b}{l^2} = \frac{Wa^2(l-a)}{l^2} - \frac{W((\delta a+a)^2)(l-(a+\delta a))}{l^2}$

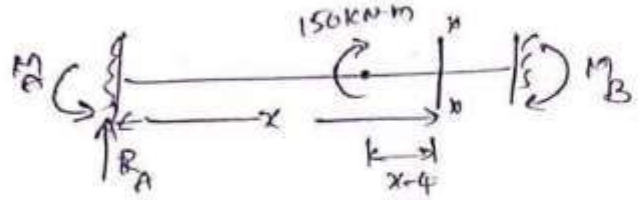
$= \frac{Wa^2(l-a)}{l^2} - \frac{W(a^2 + 2a\delta a + \delta a^2)(l-a-\delta a)}{l^2}$

$= \frac{Wa^2(l-a)}{l^2} + \frac{Wl(a+\delta a)^3}{l^3} = \frac{M}{l^2}[a(2l-3a)]$

Prob 6 For the given beam find (a) Moment at fixed ends.
(b) Reactions (c) SFD & BMD



Sol:



Bending moment at section x-x = $M_x = R_A x - M_A + 150(x-4)^0$

but $M = EI\frac{d^2y}{dx^2}$

$$EI\frac{d^2y}{dx^2} = R_A x - M_A + 150(x-4)^0$$

$$EI\frac{dy}{dx} = \frac{R_A x^2}{2} - M_A x + C_1 + 150(x-4)^1$$

At $x=0$, $\frac{dy}{dx}=0 \Rightarrow C_1 = 0$.

$$EI\frac{dy}{dx} = \frac{R_A x^2}{2} - M_A x + 150(x-4) \longrightarrow (1)$$

$$EIy = \frac{R_A x^3}{6} - \frac{M_A x^2}{2} + C_2 + 150\frac{(x-4)^2}{2}$$

At $x=0$, $y=0$, $\Rightarrow C_2 = 0$.

$$EIy = \frac{R_A x^3}{6} - \frac{M_A x^2}{2} + \frac{150(x-4)^2}{2} \longrightarrow (2)$$

At $x=6m$, $\frac{dy}{dx}=0$ from eq (1)

$$0 = \frac{R_A(6)^2}{2} - M_A(6) + 150(6-4)$$

$$0 = \frac{R_A \times 6 \times 6}{2} - 6M_A + 150(2) \Rightarrow 18R_A - 6M_A + 300 = 0$$

$$18R_A - 6M_A = -300 \longrightarrow (3)$$

At $x=6m$, $y=0$, from eq (2).

$$0 = \frac{R_A(6^3)}{6} - \frac{M_A(6)^2}{2} + \frac{150(6-4)^2}{2}$$

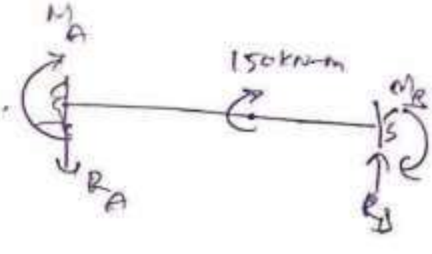
$$0 = 36R_A - 18M_A + 300 \Rightarrow 36R_A - 18M_A = -300 \longrightarrow (4)$$

$R_A = \frac{-100}{3}$ KN, $M_A = -50$ KN-m.

$\Sigma V = 0 \Rightarrow R_A + R_B = 0$.

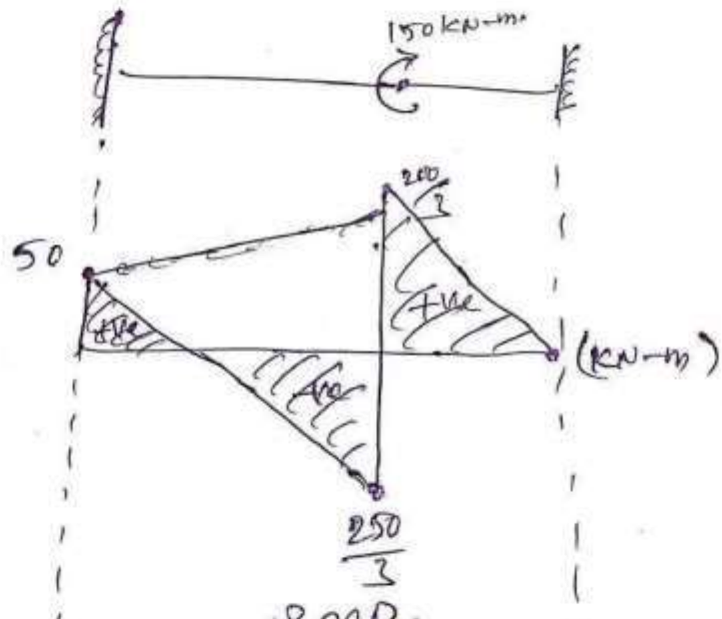
$R_B = \frac{100}{3}$ KN.

$\Sigma M_A = 0 \Rightarrow -R_B \times 6 + M_B + 150 + M_A = 0$.

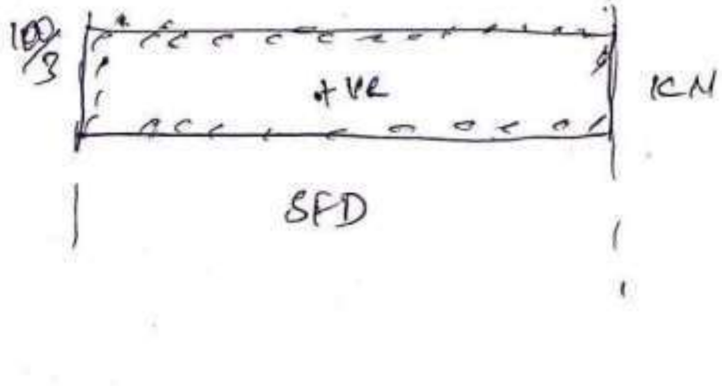


$\frac{-100}{3} \times 6 + M_B + 150 + 50 = 0$

$M_B = 0$

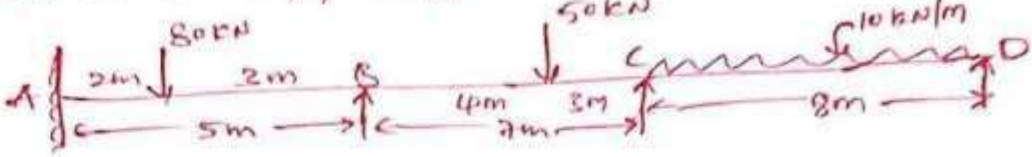


BMD.



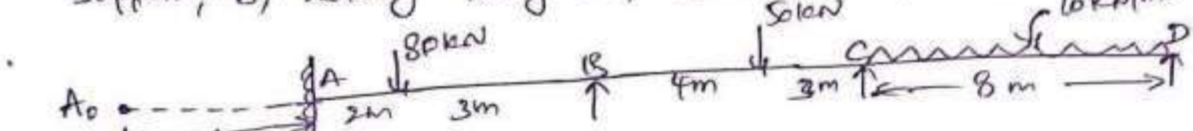
SFD

Prob: Solve the given continuous beam as shown in fig. If the end 'A' is fixed & end 'D' is simply supported.



Sol:

To handle fixed end 'A' case at end A, end 'A' is made continuous support, by adding imaginary beam A_0A span l_0.



Step ①: ~~Fixed End mom.~~ Free Bending moment

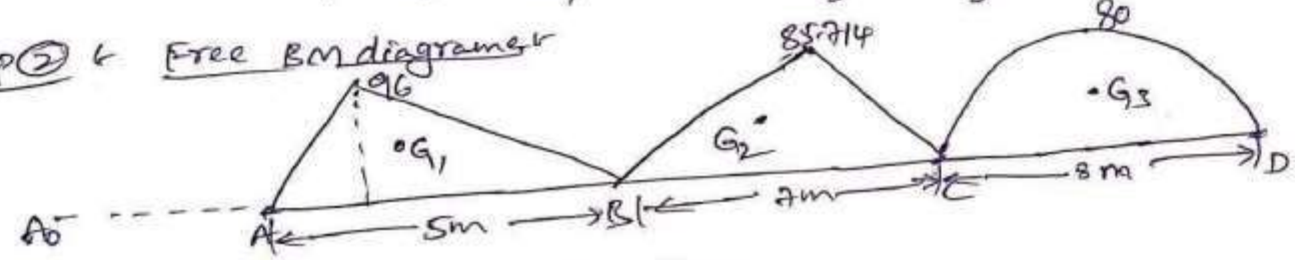
For span $A_0A = 0$,

For span AB, The BM $= \frac{Wab}{l} = \frac{80\times2\times3}{5} = 96$ kN-m.

For span BC, The BM $= \frac{Wab}{l} = \frac{50\times4\times3}{7} = 85.714$ kN-m

For span CD, the BM $= \frac{wl^2}{8} = \frac{10\times8^2}{8} = 80$ kN-m

Step ②: Free BM diagram:



For span A_0A: $A = 0$, $\bar{x} = 0$.

For span AB: $A = \frac{1}{2}\times5\times96 = 240$, $x_l = \frac{a+l}{3} = \frac{2+5}{3} = 2.3$ m.

$x_R = \frac{b+l}{3} = \frac{3+5}{3} = 2.6$ m.

For span BC: $A = \frac{1}{2}\times7\times85.714 = 299.99 \approx 300$

$x_l = \frac{a+l}{3} = \frac{4+7}{3} = 3.66$ m, $x_R = \frac{b+l}{3} = \frac{3+7}{3} = 3.33$ m

For span CD: $A = \frac{2}{3}\times8\times80 = 426.6$,

$x_l = x_R = \frac{l}{2} = \frac{8}{2} = 4$ m.

Step ③: Apply Three moment Theorem:

For span A_0A & AB:

$$M_{A_0}l_0 + 2M_A(l_0+l_1) + M_B(l_1) = \frac{-6A_0\bar{x}_0}{l_0} - \frac{6A_1\bar{x}_1}{l_1}$$

$$0 + 2M_A(0+5) + M_B(5) = 0 - \frac{6\times240\times2.66}{5}$$

$$10M_A + 5M_B = -766.08 \Rightarrow 2M_A + M_B = -153.216 \rightarrow ①$$

For span AB & BC :-

$$M_A l_1 + 2M_B(l_1+l_2) + M_C l_2 = -\frac{6A_1\bar{x}_1}{l_1} - \frac{6A_2\bar{x}_2}{l_2}$$

$$M_A(5) + 2M_B(5+7) + M_C(7) = -\frac{6\times 240\times 2.33}{5} - \frac{6\times 300\times 3.33}{7}$$

$$5M_A + 24M_B + 7M_C = -1527.32 \rightarrow (2)$$

For span BC & CD :-

$$M_B l_1 + 2M_C(l_1+l_2) + M_D(l_2) = \frac{-6A_1\bar{x}_1}{l_1} - \frac{6A_2\bar{x}_2}{l_2}$$

$$M_B(7) + 2M_C(7+8) + 0 = -\frac{6\times 300\times 3.66}{7} - \frac{6\times 426.6\times 4}{8}$$

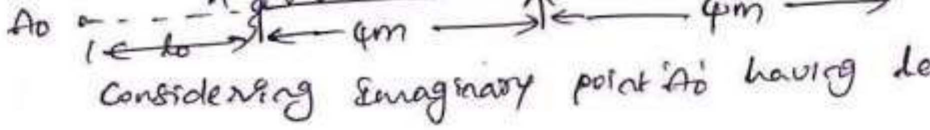
$$7M_B + 30M_C = -2220.94 \longrightarrow (3)$$

From eq(1) & eq(2) & eq(3)

$M_A = 61.002$ kN-m.
$M_B = 31.567$ kN-m.
$M_C = 66.701$ kN-m.
$M_D = 0$.

Prob :- A Continous beam ABC of uniform section with AB & BC as 4m each. is fixed at 'A' & simply support at 'B' & 'C'. The beam is carrying a uniformly distributed load of 6kN/m run throughout it's length. find support moments & reactions using three moment theorem. Also draw SFD & BMD.

Sol :-



Considering imaginary point 'A0' having length l_0 from support 'A'.

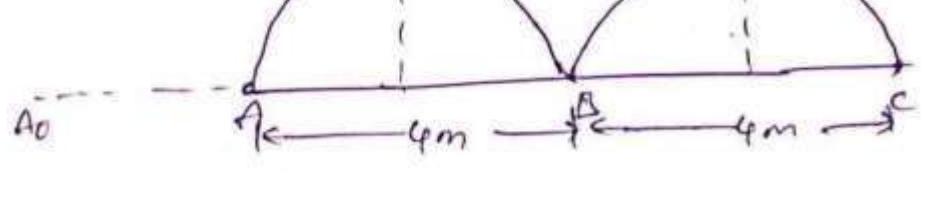
Step① :- Free Bending moment :-

For span $A_0A = 0$.

For span $AB = \frac{wl^2}{8} = \frac{6\times 4^2}{8} = 12$ kN-m.

For span $BC = \frac{wl^2}{8} = \frac{6\times 4^2}{8} = 12$ kN-m

Step② :- Draw free Bending Moment diagram :-



For span A_0A : $A = 0$, $\bar{x} = 0$.

For span AB : $A = \frac{2}{3} \times 12 \times 4 = 32$, $x_l = x_R = \frac{4}{2} = 2m$.

For span BC : $A = \frac{2}{3} \times 12 \times 4 = 32$, $x_l = x_R = \frac{4}{2} = 2m$.

Step ③ : Apply three moment theorem

For span A_0A & AB :

$$M_{A_0} + 2M_A(l_0 + l_1) + M_B(l_1) = \frac{-6A_0\bar{x}_0}{l_0} - \frac{6A_1\bar{x}_1}{l_1}$$

$$0 + 2M_A(0+4) + M_B(4) = 0 - \frac{6 \times 32 \times 2}{4}$$

$$8M_A + 4M_B = -96$$

$$2M_A + M_B = -24 \longrightarrow ①$$

For span AB & BC : support 'C' is simple support, $M_C = 0$.

$$M_A l_1 + 2M_B(l_1 + l_2) + M_C l_2 = \frac{-6A_1\bar{x}_1}{l_1} - \frac{6A_2\bar{x}_2}{l_2}$$

$$4M_A + 2M_B(4+4) + 0 = -96 - 96$$

$$4M_A + 16M_B = -192$$

$$M_A + 4M_B = -48 \longrightarrow ②$$

From eq ① & eq ②

$$2(-48 - 4M_B) + M_B = -24$$

$$-96 - 8M_B + M_B = -24$$

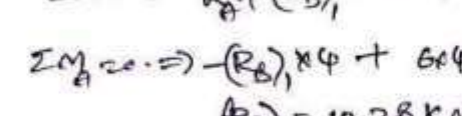
$$-7M_B = 72 \Rightarrow M_B = -10.28\ kN\text{-}m.$$

$M_A = -6.88$ kN-m.

Final moments $M_A = -6.88$ kN-m, $M_B = -10.28$ kN-m, $M_C = 0$.

Reactions :

For span AB :



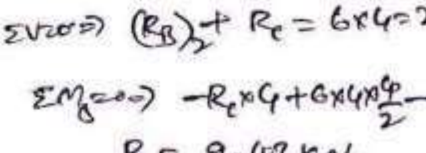
$\Sigma V = 0 \Rightarrow R_A + (R_B)_1 = 6 \times 4 = 24$

$\Sigma M_A = 0 \Rightarrow -(R_B)_1 \times 4 + 6 \times 4 \times \frac{4}{2} - M_A = 0.$

$(R_B)_1 = 10.28$ kN.

$R_A = 13.72$ kN.

For span BC :



$\Sigma V = 0 \Rightarrow (R_B)_2 + R_C = 6 \times 4 = 24$

$\Sigma M_B = 0 \Rightarrow -R_C \times 4 + 6 \times 4 \times \frac{4}{2} - M_B = 0$

$R_C = 9.43$ kN.

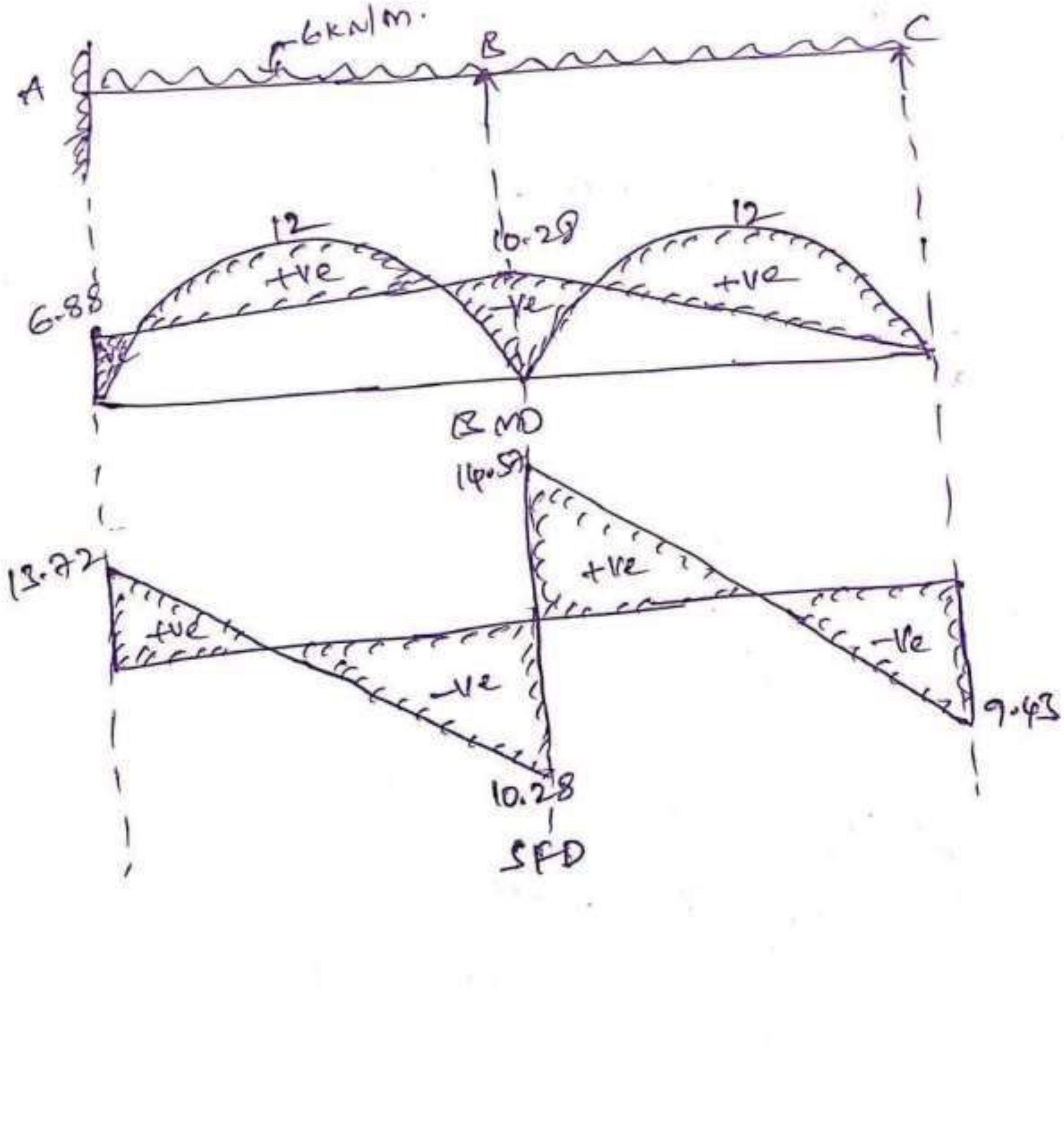
$(R_B)_2 = 14.57$ kN.

Final reactions.

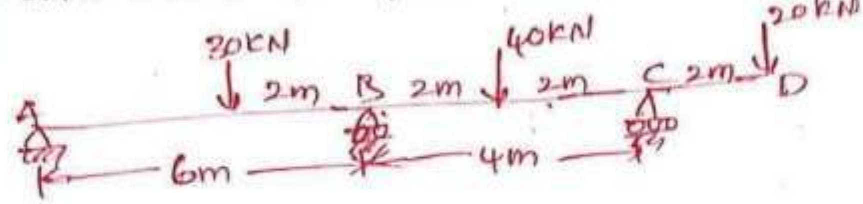
$R_A = 13.72 kN.$

$R_B = (R_B)_1 + (R_B)_2 = 10.28 + 14.57 = 24.85 kN.$

$R_C = 9.43 kN.$



Prob: Analyse the Continuous beam ABCD as shown in fig. If support 'C' Settles down by 5mm. Take $E = 15\ kN/mm^2$ & MOI is constant throughout $I = 5\times10^9\ mm^4$.



Sol:-

Step ①:- Free Bending moments:

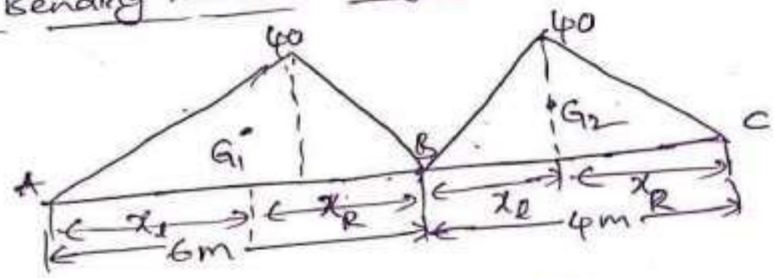
For span AB = $\frac{Wab}{l} = \frac{30\times4\times2}{6} = 40$ kN-m.

For span BC = $\frac{Wl}{4} = \frac{40\times4}{4} = 40$ kN-m.

For span CD = $Wl = 20\times2 = 40$ kN-m $= M_C$

$M_A = 0$, since it is simply supported.

Step ②:- Free Bending moment diagrams:



For span AB: $A_1 = \frac{1}{2}\times6\times40 = 120$

$x_l = \frac{a+l}{3} = \frac{4+6}{3} = \frac{10}{3}$ & $x_R = \frac{b+l}{3} = \frac{2+6}{3} = 8/3$

For span BC: $A_2 = \frac{1}{2}\times4\times40 = 80$

$x_l = x_R = \frac{l}{2} = \frac{4}{2} = 2m.$

$M_C = -40$ kN-m, $\delta_C = -5mm = -0.005m.$

$E = 15\ kN/mm^2 = 15\times10^6\ kN/m^2$

$I = 5\times10^9\ mm^4 = 5\times10^{-3}\ m^4$

Step ③:- Apply three moment theorem:

For span AB & BC: $M_A l_1 + 2M_B(l_1+l_2) + M_C l_2 = -\frac{6A_1\bar{x}_1}{l_1} - \frac{6A_2\bar{x}_2}{l_2} + \frac{6EI}{l_2}\left(\frac{\delta_1}{l_1}+\frac{\delta_2}{l_2}\right)$

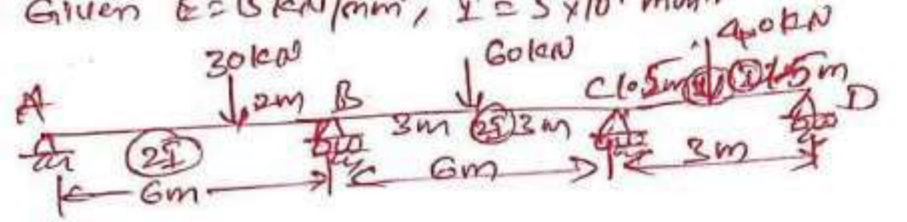
$0 + 2M_B(6+4) + (-40\times4) = -\frac{6\times120\times10/3}{6} - \frac{6\times80\times2}{4} + \frac{6\times15\times10^6\times5\times10^{-3}(-0.005)}{4}$

$2M_B\times10 = -1042.5$

$M_B = -52.125$ kN-m.

∴ Final moments $M_A = 0$, $M_B = -52.125$ kN-m & $M_C = -40$ kN-m.

Prob:- Analyse the Continuous beam ABCD shown in fig. If support 'C' sinks by 5mm, Given $E = 15\ kN/mm^2$, $I = 5\times10^9\ mm^4$.



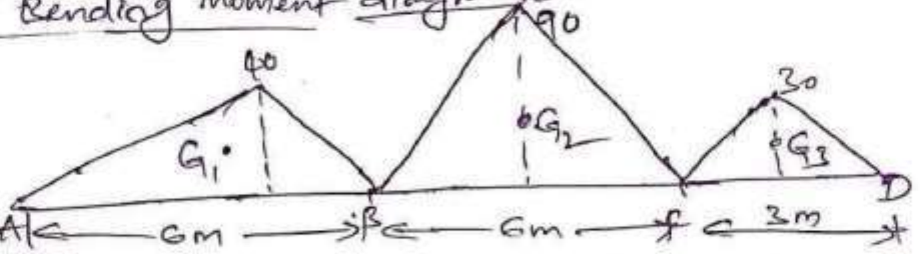
Sol:- Step①:- Free Bending Moment

For span AB, $BM = \frac{Wab}{l} = \frac{30\times4\times2}{6} = 40\ kN\text{-}m.$

For span BC, $BM = \frac{Wl}{4} = \frac{60\times6}{4} = 90\ kN\text{-}m.$

For span CD, $BM = \frac{Wl}{4} = \frac{40\times3}{4} = 30\ kN\text{-}m.$

Step②:- Free Bending moment diagram:-



For Span AB:- $A = \frac{1}{2}\times6\times40 = 120$,

$\bar{x}_l = \frac{a+l}{3} = \frac{4+6}{3} = 10/3$, $\bar{x}_r = \frac{b+l}{3} = \frac{2+6}{3} = 8/3$

For Span BC:- $A = \frac{1}{2}\times6\times90 = 270$,

$\bar{x}_l = \bar{x}_r = \frac{l}{2} = \frac{6}{2} = 3m.$

For span CD:- $A = \frac{1}{2}\times3\times30 = 45$,

$\bar{x}_l = \bar{x}_r = \frac{3}{2} = 1.5m.$

$\delta_C = 5mm = 0.005m$

$E = 12\ kN/mm^2 = 12\times10^6\ kN/m^2$

$I = 5\times10^9\ mm^4 = 5\times10^{-3}\ m^4.$

Step③:- Apply theorem of three moments:-

For span A-B-C:- We know support A & D is simple supports

$M_A = M_D = 0.$ $\delta_1 = 0$, $\delta_2 = -0.005$

$$M_A\frac{l_1}{I} + 2M_B\left(\frac{l_1}{I_1}+\frac{l_2}{I_2}\right) + M_C\left(\frac{l_2}{I_2}\right) = -\frac{6A_1\bar{x}_1}{l_1 I_1} - \frac{6A_2\bar{x}_2}{l_2 I_2} + \frac{6E\delta_1}{l_1} + \frac{6E\delta_2}{l_2}$$

$$0 + 2M_B\left(\frac{6}{2I}+\frac{6}{2I}\right) + M_C\left(\frac{6}{2I}\right) = \frac{-6\times120\times10/3}{6(2I)} - \frac{6\times270\times3}{6(2I)} + 0 + \frac{6\times12\times10^6\times(-0.005)}{6}$$

$$\frac{12M_B}{I} + \frac{3M_C}{I} = -\frac{160}{I} - \frac{145}{I} - \left(\frac{6\times15\times10^6\times(0.005)}{6}\right)$$

$$12M_B + 3M_C = -160 - 405 - \left(\frac{6\times15\times10^6\times0.005\times5\times10^{-3}}{6}\right)$$

$$12M_B + 3M_C = -160 - 405 - 375$$

$$12M_B + 3M_C = -940 \longrightarrow ①$$

$\delta_1 = 0.005m, \quad \delta_2 = 0.005m.$

For span B-C-D :-

$$M_B\left(\frac{l_1}{I_1}\right) + 2M_C\left(\frac{l_1}{I_1} + \frac{l_2}{I_2}\right) + M_D\left(\frac{l_2}{I_2}\right) = \frac{-6A_1\bar{x}_1}{l_1 I_1} - \frac{6A_2\bar{x}_2}{l_2 I_2} + \frac{6EI\delta_1}{l_1 I_1} + \frac{6EI\delta}{l_2 I_2}$$

$$M_B\left(\frac{6}{2I}\right) + 2M_C\left(\frac{6}{2I} + \frac{3}{I}\right) + 0 = \frac{-6\times270\times3}{6\times(2I)} - \frac{6\times45\times1.5}{3\times I} + \frac{6\times12\times10^6(0.005)}{6} + \left(\frac{6\times15\times10^6\times0.005}{3}\right)$$

$$\frac{3M_B}{I} + \frac{12M_C}{I} = \frac{-405}{I} - \frac{135}{I} + \left(\frac{6\times12\times10^6(0.005)}{6}\right) + \left(\frac{6\times15\times10^6\times0.005}{3}\right)$$

$$3M_B + 12M_C = -405 - 135 + \left(\frac{6\times12\times10^6\times0.005\times5\times10^{-3}}{6}\right) + \left(\frac{6\times15\times10^6\times0.005\times5\times10^{-3}}{3}\right)$$

$$3M_B + 12M_C = -405 - 135 + 375 + 750$$

$$3M_B + 12M_C = 585 \longrightarrow ②$$

From eq ① & eq ②

$M_B = -96.556$ kN-m,

$M_C = -72.889$ kN-m.

Final moments

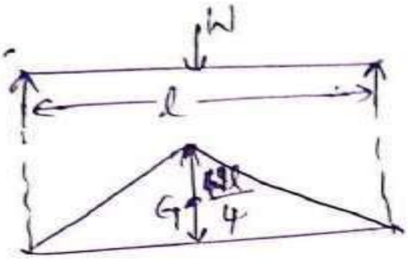
$M_A = 0,$

$M_B = -96.556$ kN-m.

$M_C = -72.889$ kN-m.

$M_D = 0.$

①

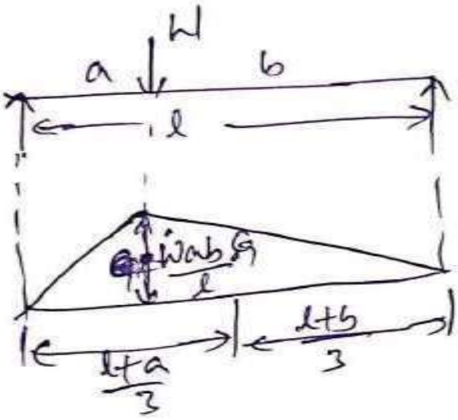


Max BM = $\frac{Wl}{4}$,

Area $A = \frac{1}{2} \times l \times \frac{Wl}{4}$,

CG from left & right = $\frac{l}{2}$

②



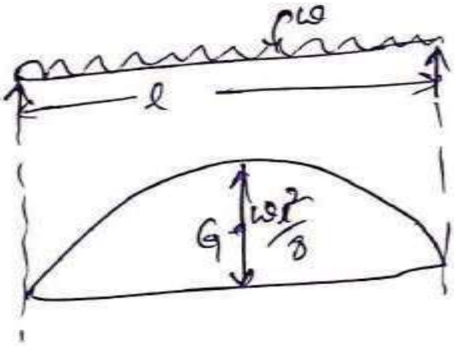
Max BM = $\frac{Wab}{l}$,

Area $A = \frac{1}{2} \times l \times \frac{Wab}{l}$

CG from left support = $\frac{l+a}{3}$

CG from right support = $\frac{l+b}{3}$

③



Max BM = $wl^2/8$

Area (A) = $\frac{2}{3} \times l \times \frac{wl^2}{8}$

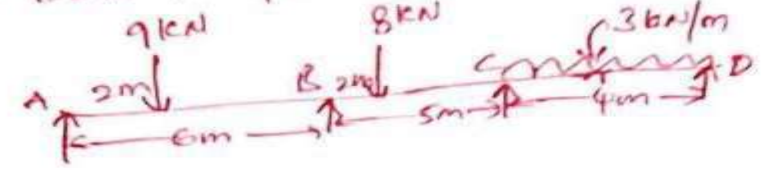
CG from left & right = $\frac{l}{2}$

~~clapeyron.~~

clapeyron's theorem:

$$M_A l_1 + 2M_B (l_1 + l_2) + M_C l_2 = -\left(\frac{6A_1\bar{x}_1}{l_1} + \frac{6A_2\bar{x}_2}{l_2}\right)$$

prob: Analyse the continuous beam ABCD shown in fig. using theorem of Three moments. Draw SFD & BMD



Sol: Step ①: Free Bending Moment:

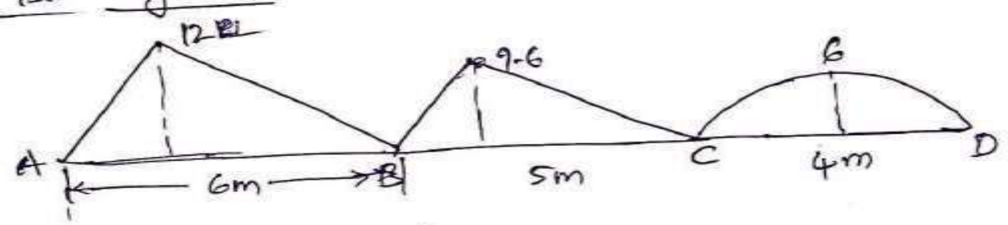
Considering each span simply supported.

for span AB, $BM = \frac{Wab}{l} = \frac{9\times2\times4}{6} = 12\,KN\text{-}m.$

for span BC, $BM = \frac{Wab}{l} = \frac{8\times2\times3}{5} = 9.6\,KN\text{-}m.$

for span CD, $BM = \frac{wl^2}{8} = \frac{3\times4^2}{8} = 6\,KN\text{-}m.$

Step ②: Free BM diagram:



For Span AB: Area $A = \frac{1}{2}\times b\times h = \frac{1}{2}\times12\times6 = 36.$

$x_l = \frac{a+l}{3} = \frac{2+6}{3} = 2.67\,m,\quad x_R = \frac{b+l}{3} = \frac{4+6}{3} = 3.33\,m$

For span BC: $Area\ (A) = \frac{1}{2}\times b\times h = \frac{1}{2}\times9.6\times5 = 24$

$x_l = \frac{a+l}{3} = \frac{2+5}{3} = 2.33\,m \quad x_R = \frac{b+l}{3} = \frac{3+5}{3} = 2.66\,m.$

For span CD: $Area\ (A) = \frac{2}{3}\times base\times height = \frac{2}{3}\times4\times6 = 16$

$x_l = x_R = \frac{l}{2} = \frac{4}{2} = 2\,m.$

Step ③: Apply three moment Theorem:

For span A-B-C:

$$M_A l_1 + 2M_B(l_1+l_2) + M_C l_2 = -\left(\frac{6A_1\bar{x}_1}{l_1} + \frac{6A_2\bar{x}_2}{l_2}\right)$$

$$0 + 2M_B(6+5) + M_C(5) = -\left(\frac{6\times36\times2.67}{6} + \frac{6\times24\times2.66}{5}\right)$$

$$22M_B + 5M_C = -172.728 \rightarrow ①$$

$$2M_B(11) = -(172.728) \Rightarrow 22M_B = -172.728$$

For span B–C–D:

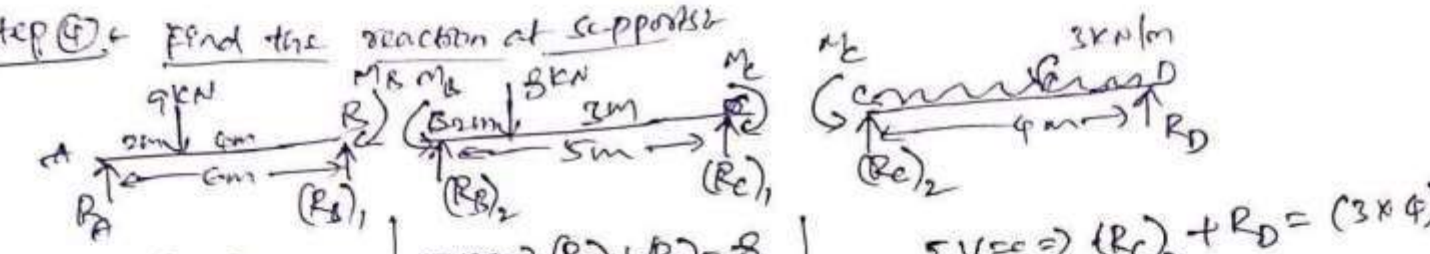
$$M_B l_1 + 2M_C(l_1 + l_2) + M_D(l_2) = -\left(\frac{6A_1\bar{x}_1}{l_1} + \frac{6A_2\bar{x}_2}{l_2}\right)$$

$$M_B(5) + 2M_C(5+4) + 0 = -\left(\frac{6\times 24\times 2.33}{5} + \frac{6\times 16\times 2}{4}\right)$$

$$5M_B + 18M_C = -115.10 \longrightarrow (2)$$

From eq (1) & eq (2), $M_B = -6.83$ kN·m & $M_C = -4.497$ kN·m.

Step (4): Find the reaction at supports:



$\Sigma V = 0 \Rightarrow R_A + (R_B)_1 = 9$

$\Sigma M_A = 0 \Rightarrow 9\times 2 - (R_B)_1\times 6 + 6.83 = 0.$

$(R_B)_1 = 4.138$ KN.

$R_A = 4.862$ KN.

$\Sigma V = 0 \Rightarrow (R_B)_2 + (R_C)_1 = 8$

$\Sigma M_B = 0 \Rightarrow$

$-M_B + 8\times 2 - (R_C)_1\times 5 + M_C = 0.$

$-6.83 + 16 - (R_C)_1\times 5 + 4.497 = 0$

$(R_C)_1 = 2.733$ kN

$(R_B)_2 = 5.267$ kN.

$\Sigma V = 0 \Rightarrow (R_C)_2 + R_D = (3\times 4)$

$\Sigma M_C = 0 \Rightarrow$

$-M_C + 3\times 4\times 2 - R_D\times 4 = 0.$

$-4.497 + 24 - R_D\times 4 = 0.$

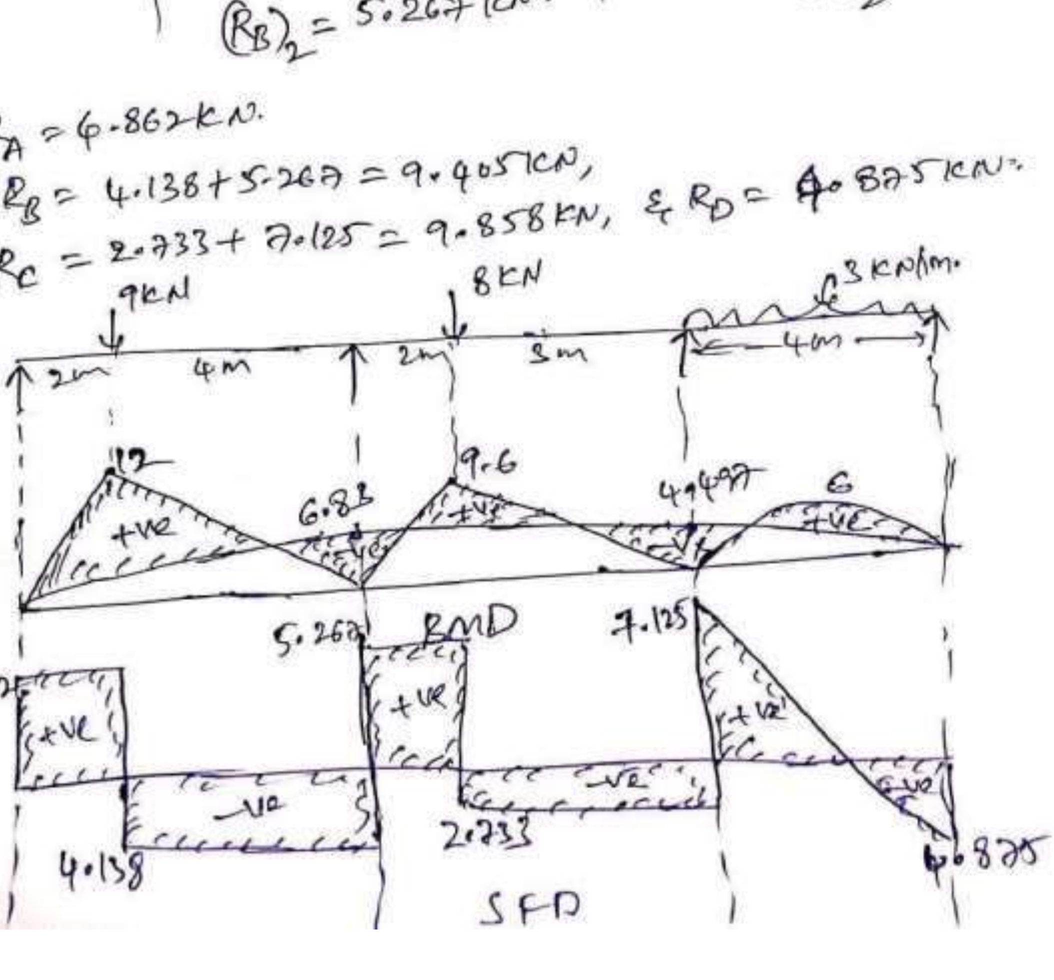
$R_D = 4.875$ KN

$(R_C)_2 = 7.125$ KN.

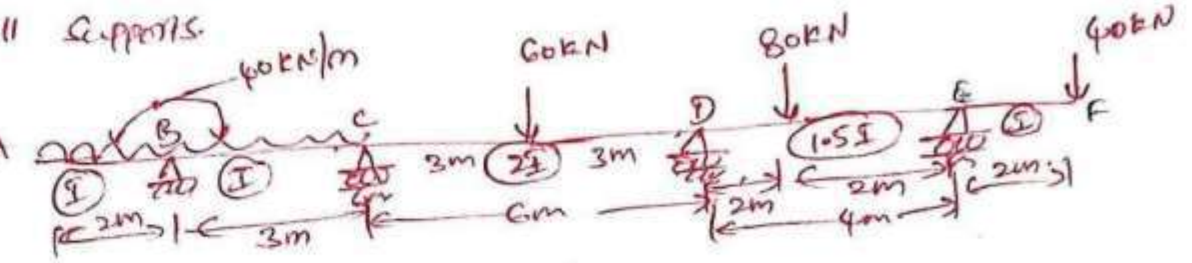
$\therefore R_A = 4.862$ KN.

$R_B = 4.138 + 5.267 = 9.405$ kN,

$R_C = 2.733 + 7.125 = 9.858$ KN, & $R_D = 4.875$ kN.



Prob :- Analyse the Continuous beam as shown in fig. & determine the moment at all supports.



Sol:- Step① :- Free Moment :-

for span AB = $w\frac{l^2}{2} = 40\times 2\times\frac{2}{2} = 80$ kN-m. $= M_B$

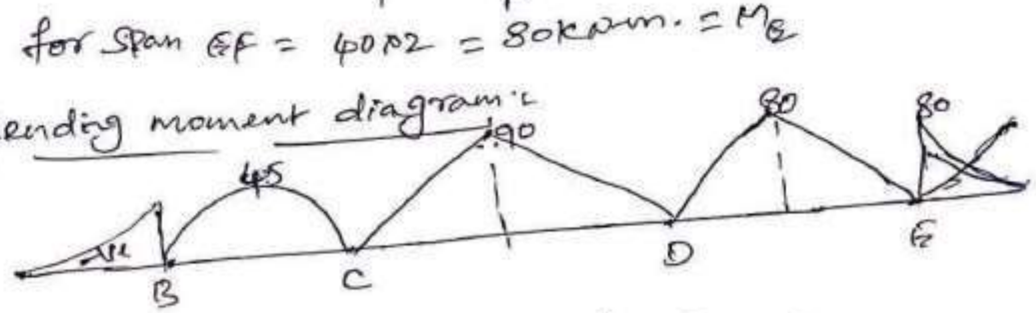
for span BC = $\frac{wl^2}{8} = \frac{40\times 3^2}{8} = 45$ kN-m.

for span CD = $\frac{Wl}{4} = \frac{60\times 6}{4} = 90$ kN-m.

for span DE = $\frac{Wl}{4} = \frac{80\times 4}{4} = 80$ kN-m.

for span EF = $40\times 2 = 80$ kN-m. $= M_E$

Step② :- Free Bending moment diagram :-



For span BC :- $(A) = \frac{2}{3}\times 3\times 45 = 90$, $x_l = x_R = \frac{l}{2} = \frac{3}{2} = 1.5$m.

For span CD, $A = \frac{1}{2}\times 6\times 90 = 270$, $x_l = x_R = \frac{l}{2} = \frac{6}{2} = 3$m.

For span DE :- $A = \frac{1}{2}\times 4\times 80 = 160$, $x_l = x_R = \frac{l}{2} = \frac{4}{2} = 2$m.

Step③ :- Apply three moment theorem :-

For span B-C-D :-

$$M_B\left(\frac{l_1}{I}\right) + 2M_C\left(\frac{l_1}{I}+\frac{l_2}{2I}\right) + M_D\left(\frac{l_2}{2I}\right) = -\left(\frac{6A_1\bar{x}_1}{l_1 I}\right) - \left(\frac{6A_2\bar{x}_2}{l_2 (2I)}\right)$$

$$(-80)\frac{3}{I} + 2M_C\left(\frac{3}{I}+\frac{6}{2I}\right) + M_D\left(\frac{6}{2I}\right) = -\left(\frac{6\times 90\times 1.5}{3I}\right) - \left(\frac{6\times 270\times 3}{(2I)\times 6}\right)$$

$$\frac{-240}{I} + \frac{12M_C}{I} + \frac{3M_D}{I} = \frac{-270}{I} - \frac{405}{I}$$

$$4M_C + M_D = -145 \longrightarrow ①$$

For span C - D - E

$$M_C\left(\frac{l_1}{2I}\right)+2M_D\left(\frac{l_1}{2I}+\frac{l_2}{1.5I}\right)+M_E\left(\frac{l_2}{1.5I}\right)=-\left(\frac{6A_1\bar{x}_1}{l_1(2I)}\right)-\left(\frac{6A_2\bar{x}_2}{l_2(1.5I)}\right)$$

$$M_C\left(\frac{6}{2I}\right)+2M_D\left(\frac{6}{2I}+\frac{4}{1.5I}\right)+(-80)\left(\frac{4}{1.5I}\right)=-\left(\frac{6\times 270\times 3}{2I\times 6}\right)-\left(\frac{6\times 160\times 2}{1.5I\times 4}\right)$$

$$\frac{3M_C}{I}+\frac{11.33M_D}{I}-\frac{213.33}{I}=-\frac{405}{I}-\frac{320}{I}$$

$$3M_C+11.33M_D-213.33=-405-320$$

$$3M_C+11.33M_D=-511.66 \longrightarrow (2)$$

From eq (1) & eq (2)

$$M_C=-26.732\,kN\text{-}m.$$

$$M_D=-38.071\,kN\text{-}m.$$

Final moment $M_B=-80\,kN\text{-}m.$

$$M_C=-26.732\,kN\text{-}m.$$

$$M_D=-38.071\,kN\text{-}m$$

$$M_E=-80\,kN\text{-}m.$$

UNIT-IV

SLOPE DEFLECTION METHOD

Continuous beams and rigid frames (with and without sway) – Symmetry and antisymmetry – Simplification for hinged end – Support displacements

Introduction:

- This method was first proposed by Prof. George A. Maney in 1915.
- It is ideally suited to the analysis of continuous beams and rigid jointed frames.
- Basic unknowns like slopes and deflections of joints are found out.
- Moments at the ends of a member is first written down in terms of unknown slopes and deflections of end joints.
- Considering the joint equilibrium conditions, a set of equations are formed and solutions of these simultaneous equations gives unknown slopes and deflections.
- Then end moments of individual members are determined.
- It involves solutions of simultaneous equations; a problem with more than three unknowns is considered a difficult problem for hand calculations. Hence this method was sidelined by moment distribution method with the help of computers; solutions for any number of simultaneous equations can be obtained early.
- The development of this method in the matrix form is "Stiffness Matrix Method" (it is commonly used for the analysis of large structures with the help of computers.

Assumptions made in slope-deflection method

- All joints are rigid.
- The rotations of joints are treated as unknowns.
- Between each pair of the supports the beam section is constant.
- The joint in structure may rotate or deflect as a whole, but the angles between the members meeting at that joint remain the same.
- Distortions due to axial deformations are neglected.
- Shear deformations are neglected.

Sign Conventions:

Moments:

- Clockwise moments = $(+)^{ive}$
- Anti-clockwise moments = $(-)^{ive}$

Rotations:

- Clockwise rotations = $(+)^{ive}$
- Anti-clockwise rotations = $(-)^{ive}$

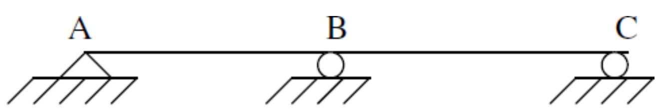
Settlements:

- Right side support is below left side support = $(+)^{ive}$
- Left side support is below right side support = $(-)^{ive}$

<u>Applications of Slope Deflection Equations:</u>

- Rigid jointed structures can be analyzed.
- Continuous Beams
- Frames without side sway (Non-Sway)
- Frames with side sway (Sway)

<u>The beam shown in Fig. is to be analyzed by slope-deflection method. What are the unknowns and, to determine them, what are the conditions used?</u>



Unknowns: θ_A, θ_B, θ_C

Equilibrium equations used: (i) MAB = 0 (ii) MBA + MBC = 0 (iii) MCB = 0

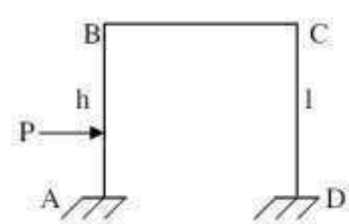
<u>Write down the slope deflection equation for a fixed end support.</u>

The slope deflection equation for end A is $M_{AB} = M'_{AB} + \frac{2EI}{l}\left[2\theta_A + \theta_B + \frac{3\Delta}{l}\right]$

Here $\theta_A = 0$. Since there is no support settlement, $\Delta = 0$.

$$M_{AB} = M'_{AB} + 2EI\left[\theta_B + \underline{3\Delta}\right]$$

<u>Write down the equilibrium equations for the frame shown in Fig.</u>



Unknowns : θ_B , θ_C

Equilibrium equations : At B, $M_{BA} + M_{BC} = 0$

At C, $M_{CB} + M_{CD} = 0$

Shear equation : $\frac{M_{AB} + M_{BA} - Ph}{l} + \frac{M_{CD} + M_{DC}}{l} + P = 0$

<u>Limitations of slope deflection method</u>

- It is not easy to account for varying member sections
- It becomes very cumbersome when the unknown displacements are large in number.

Why slope-deflection method is called a 'displacement method'?

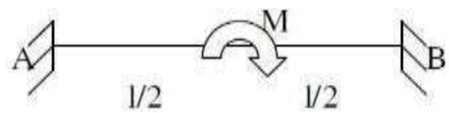
- In slope-deflection method, displacements (like slopes and displacements) are treated as unknowns and hence the method is a „displacement method".

Degrees of freedom

- In a structure, the numbers of independent joint displacements that the structure can undergoes are known as degrees of freedom.

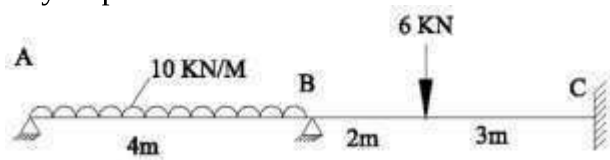
Write the fixed end moments for a beam carrying a central clockwise moment.

Fixed end moments : $M'_{AB} = M'_{BA} = \frac{M}{4}$



Problems:

1. Analyse the continuous beam given in figure by slope deflection method and draw the B.M.D&S.F.D.



Step 1: Fixed end moments

M^F_{AB}	=	$-WL^2/12$	=	$-10\times4^2/12$	=	-13.33 KNM
M^F_{BA}	=	$WL^2/12$	=	$10\times4^2/12$	=	-13.33 KNM
M^F_{BC}	=	$-Wab^2/L^2$	=	$-6\times2\times3^2/5^2$	=	-4.32 KNM
M^F_{CB}	=	Wa^2b/L^2	=	$6\times2^2\times3/5^2$	=	2.88 KNM

Step 2: Slope deflection equation

$M_{AB} = M^F_{AB}+2EI/L\,(2\theta_A+\theta_B)$

$M_{AB} = -13.33+EI\theta_A + 0.5EI\theta_B$ ------------------------1

$M_{BA} = M^F_{BA}+2EI/L\,(2\theta_B+\theta_A)$

$M_{BA} = 13.33+0.5EI\theta_A + EI\,\theta_B$ ------------------------2

$M_{BC} = M^F_{BC}+2EI/L\,(2\theta_B+\theta_C)$

$M_{BC} = -4.32+0.8EI\,\theta_B$ --------3

$M_{CB} = 2.88+0.4EI\,\theta_B$ --------4

Apply equilibrium conditions

$M_{AB} = 0$

$EI\,\theta_A + 0.5EI\,\theta_B = 13.33$ --------5

$M_{BA} + M_{BC} = 0$

$13.33 + 0.5EI\,\theta_A + EI\,\theta_B - 4.32 + 0.8EI\,\theta_B = 0$

$0.5EI\,\theta_A + 1.8EI\,\theta_B = -9.01$ -------------------- 6

Solve eqn 5 & 6, we get

$EI\theta_A$ = 18.39

$EI\theta_B$ = -10.11

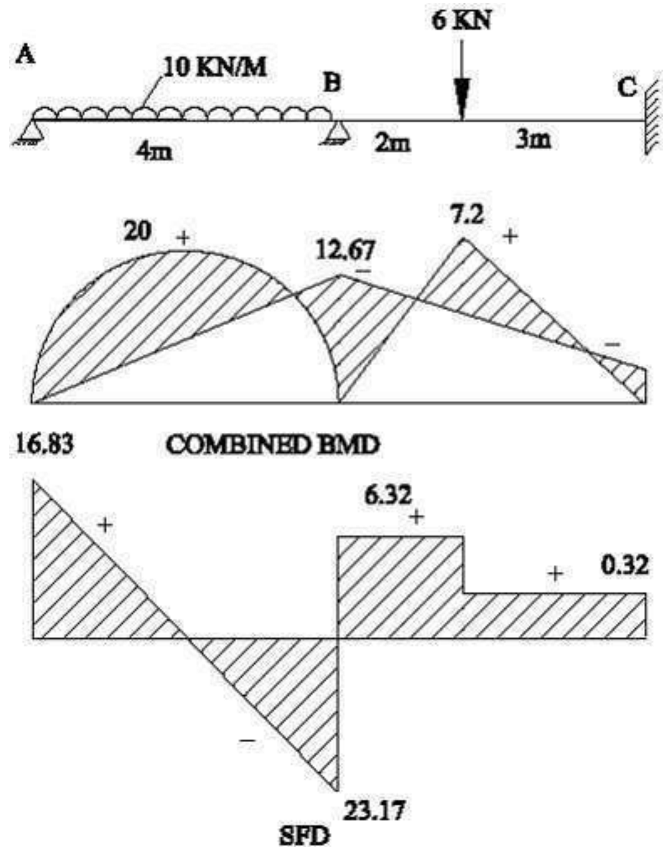
This values sub in eqn 1 to 4

M_{AB} = 0 KNM

M_{BA} = 12.67 KNM

M_{BC} = -12.67 KNM

M_{CB} = -1.16 KNM



<u>Step 3: Find the Reactions</u>

Span AB

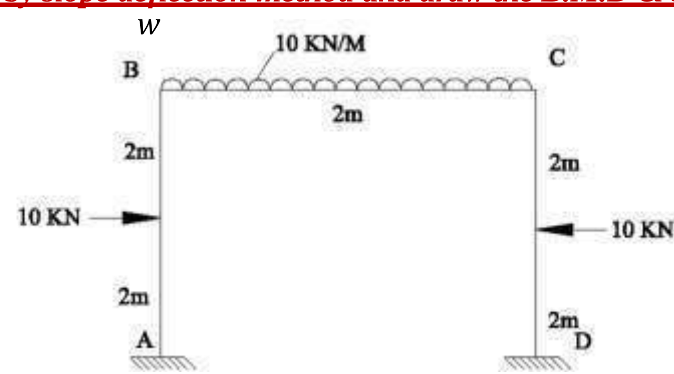
R_A = 16.83 KN

R_{B1} = 23.17 KN

R_{B2} = 6.312 KN

R_C = -0.312 KN

2. *<u>Analyze the frame given in figure by slope deflection method and draw the B.M.D & S.F.D.</u>*



<u>Step 1: fixed end moments</u>

M^F_{AB}	=	-WL/8	=	- 10×4/8	=	-5 KNM
M^F_{BA}	=	WL/8	=	10×4/8	=	5 KNM
M^F_{BC}	=	$-WL^2/12$	=	$- 10\times2^2/12$	=	- 3.33 KNM
M^F_{CB}	=	$WL^2/12$	=	$10\times2^2/12$	=	3.33 KNM
M^F_{CD}	=	-WL/8	=	- 10×4/8	=	-5 KNM
M^F_{DC}	=	WL/8	=	10×4/8	=	5 KNM

<u>Step 2: Slope deflection equation</u>

$M_{AB}=M^FAB+2EI/L(2\theta A+\theta B)$

$M_{AB}=-5 +0.5EI\theta B$ ------- 1

$M_{BA} = M^{r}_{BA} + 2EI/L\ (2\theta_B + \theta_A)$

$M_{BA} = 5 + EI\theta_B$ ------------------------------ 2

$M_{BC} = M^{F}_{BC} + 2EI/L\ (2\theta_B + \theta_C)$

$M_{BC} = -3.33 + 2EI\theta_B + EI\theta_C$ ------------------------------ 3

$M_{CB} = M^{F}_{CB} + 2EI/L\ (2\theta_C + \theta_B)$

$M_{CB} = 3.33 + 2EI\theta_C + EI\theta_B$ ------------------------------ 4

$M_{CD} = M^{F}_{CD} + 2EI/L\ (2\theta_C + \theta_D)$

$M_{CD} = -5 + EI\theta_C$ ------------------------------ 5

$M_{DC} = M^{F}_{DC} + 2EI/L\ (2\theta_D + \theta_C)$

$M_{DC} = 5 + 0.5EI\theta_C$ ------------------------------ 6

Apply equilibrium conditions

$M_{BA} + M_{BC} = 0$

$5 + EI\theta_B - 3.33 + 2EI\theta_B + EI\theta_C = 0$ --------- 7

$M_{CB} + M_{CD} = 0$

$3.33 + 2EI\theta_C + EI\theta_B - 5 + EI\theta_C = 0$ -------- 8

Solve eqn 7 & 8 we get

$EI\theta_B = -0.835$

$EI\theta_C = 0.835$

Sub this values eqn 1 to 6

$M_{AB} = -5.42$ KNM

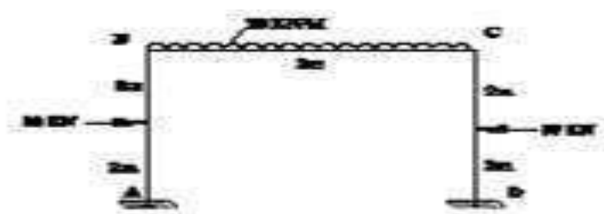
$M_{BA} = 4.17$ KNM

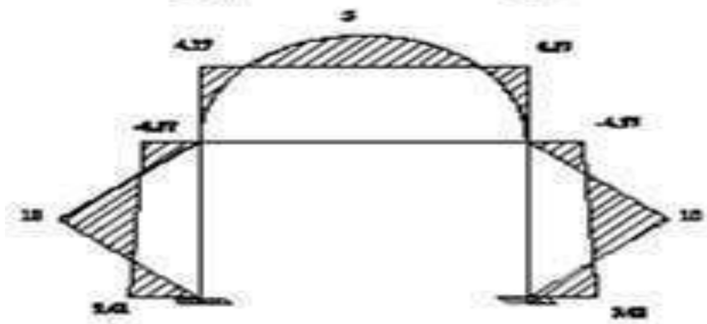
$M_{BC} = -4.17$ KNM

$M_{CB} = 4.17$ KNM

$M_{CD} = -4.17$ KNM

$M_{DC} = 5.42$ KNM





COMBINED BMD

Step 3: find reactions

Span AB:

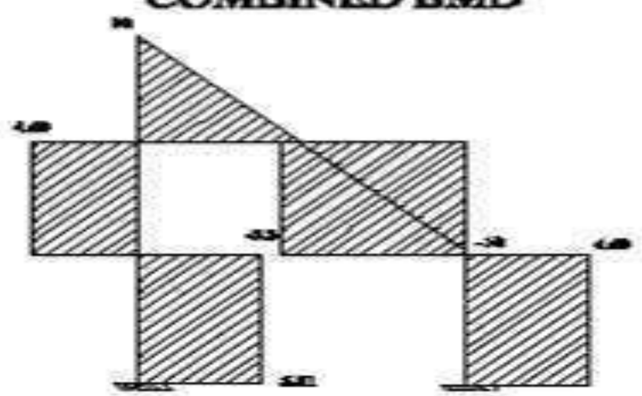
$R_A = 5.31$ KN

$R_{B1} = 4.69$ KN

Span BC:

$R_{B2} = 10$ KN

$R_{C1} = 10$ KN

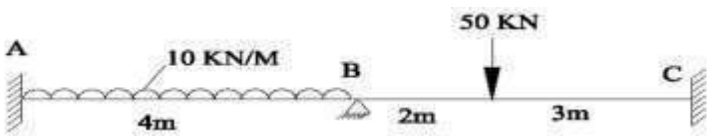


Span CD:

R_{C2} = 4.69 KN

R_D = 5.31 KN

3. ***Draw the SFD&BMD for th continuous beam shown in fig. Take $E=2\times10^5$ N/mm^2, $I=3\times10^6$ mm^4 .The support B sinks by 30 mm. Using slope deflection method.***



Step 1: fixed end moments

M^F_{AB} = $-WL^2/12$ = $-10\times4^2/12$ = -13.33 KNM

M^F_{BA} = $WL^2/12$ = $10\times4^2/12$ = -13.33 KNM

M^F_{BC} = $-Wab^2/L^2$ = $-50\times2\times3^2/5^2$ = -36 KNM

M^F_{CB} = Wa^2b/L^2 = $50\times2^2\times3/5^2$ = 24 KNM

Step 2: Slope deflection equation

M_{AB} = $M^F_{AB} + 2EI/L\,(2\theta_A+\theta_B) - 6EI\Delta/l^2$

M_{AB} = $EI\theta_B - 20$ ---------------------------------------1

M_{BA} = $M^F_{BA} + 2EI/L\,(2\theta_B+\theta_A) - 6EI\Delta/l^2$

M_{BA} = $EI\theta_B + 6.58$--------------------------------------- 2

M_{BC} = $M^F_{BC} + 2EI/L\,(2\theta_B+\theta_C) + 6EI\Delta/l^2$

M_{BC} = $0.8EI\theta_B - 31.68$-----------------------------------3

M_{CB} = $M^F_{CB} + 2EI/L\,(2\theta C+\theta_B) + 6EI\Delta/l^2$

M_{CB} = $28.32 + 0.4EI\theta_B$ ----------------------------------4

Applying equilibrium conditions

$M_{BA} + M_{BC}$ = 0

$EI\theta_B + 6.58 + 0.8EI\theta_B - 31.68$ = 0

$EI\theta_B$ = 13.94

This values sub in eqn 1 to 4

M_{AB} = -13.03 KNM

M_{BA} = 20.52 KNM

M_{BC} = -20.52 KNM

M_{CB} = 33.89KNM

Step 3: Find the reactions

Span AB

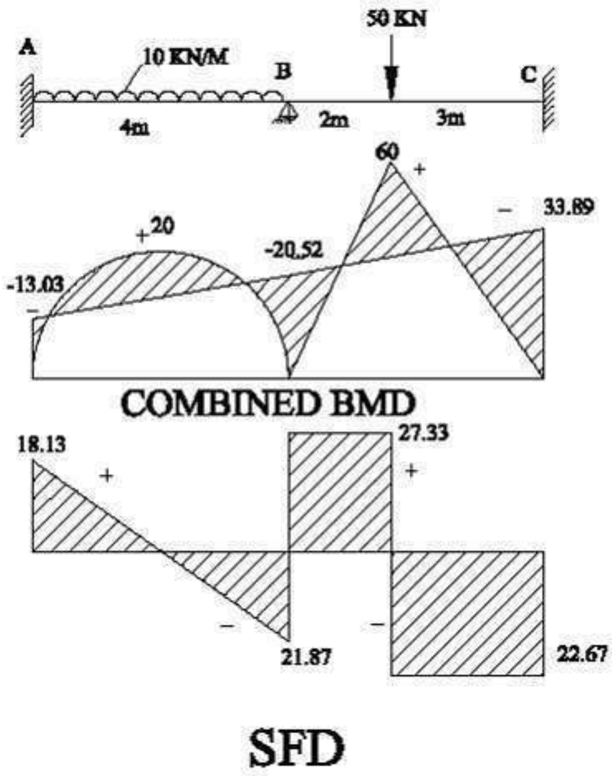
R_A = 18.13 KN

R_{B1} = 21.87 KN

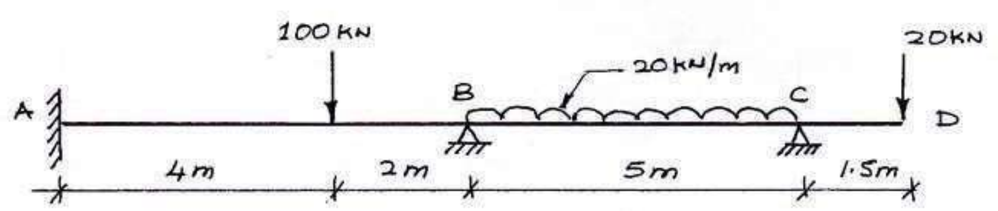
Span BC

R_{B2} = 27.33 KN

R_C = 22.67 KN



4) Analyze continuous beam ABCD by slope deflection method and then draw bending moment and SF diagram. Take EI constant.



Solution:

FEM

$$F_{AB} = -\frac{Wab^2}{L^2} = -\frac{100 \times 4 \times 2^2}{6^2} = -44.44 \text{ KNM}$$

$$F_{BA} = +\frac{Wa^2b}{L^2} = +\frac{100 \times 4^2 \times 2}{6^2} = +88.88 \text{ KNM}$$

$$F_{BC} = -\frac{wL^2}{12} = -\frac{20 \times 5^2}{12} = -41.67 \text{ KNM}$$

$$F_{CB} = +\frac{wL^2}{12} = +\frac{20 \times 5^2}{12} = +41.67 \text{ KNM}$$

$$F_{CD} = -20 \times 1.5 = -30 \text{ KNM}$$

Slope deflection equations:

$$M_{AB} = F_{AB} + \frac{2EI}{L}\left(2\theta_A + \theta_B\right) = -44.44 + \frac{1}{3}EI\theta_B \quad \text{- ----------} > (1)$$

$$M_{BA} = F_{BA} + \frac{2EI}{L}\left(2\theta_B + \theta_A\right) = +88.89 + \frac{2}{3}EI\theta_B \quad \text{------------} > (2)$$

$$M_{BC} = F_{BC} + \frac{2EI}{L}\left(2\theta_B + \theta_C\right) = -41.67 + \frac{4}{5}EI\theta_B + \frac{2}{5}EI\theta_C \quad \text{-----------} > (3)$$

$$M_{CB} = F_{CB} + \frac{2EI}{L}\left(2\theta_C + \theta_B\right) = +41.67 + \frac{4}{5}EI\theta_C + \frac{2}{5}EI\theta_B \quad \text{-----------} > (4)$$

$M_{CD} = -30$ KNM

Now, $M_{BA} + M_{BC} = 88.89 + \frac{2}{3}EI\theta_B - 41.67 + \frac{4}{5}EI\theta_B + \frac{2}{5}EI\theta_C$

:

$$= 47.22 + \frac{22}{15}EI\theta_B + \frac{2}{5}EI\theta_C = 0 \text{ --------------------------------} > (5)$$

And, $M_{CB} + M_{CD} = +41.67 + \frac{4}{5}EI\theta_C + \frac{2}{5}EI\theta_B - 30$

$$= 11.67 + \frac{2}{5}EI\theta_B + \frac{4}{5}EI\theta_C \quad ------> (6)$$

$EI\theta_B = -32.67$ Rotation@ B anticlockwise

$EI\theta_C = +1.75$ Rotation @ B clockwise

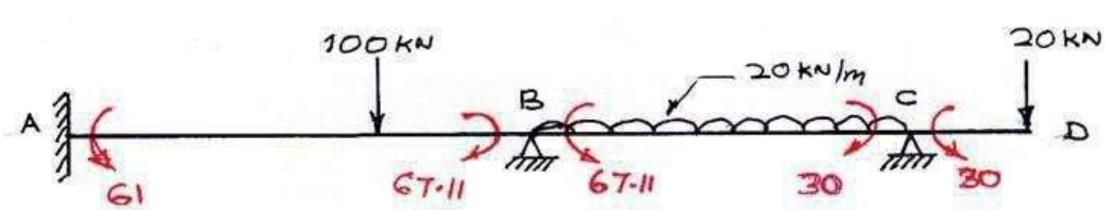
$M_{AB} = -44.44 + \frac{1}{2}(-32.67) = -61.00$ KNM

$M_{BA} = +88.89 + \frac{2}{3}(-32.67) = +67.11$ KNM

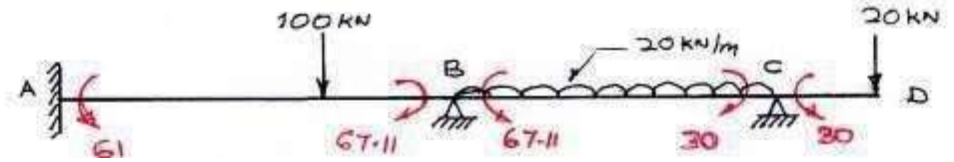
$M_{BC} = -41.67 + \frac{4}{5}(-32.67) = + \frac{2}{5}(1.75) = -67.11$ KNM

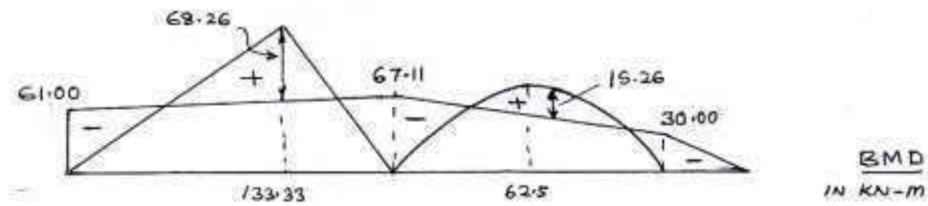
$M_{CB} = +41.67 + \frac{4}{5}(1.75) + \frac{2}{5}(-32.67) = +30.00$ KNM

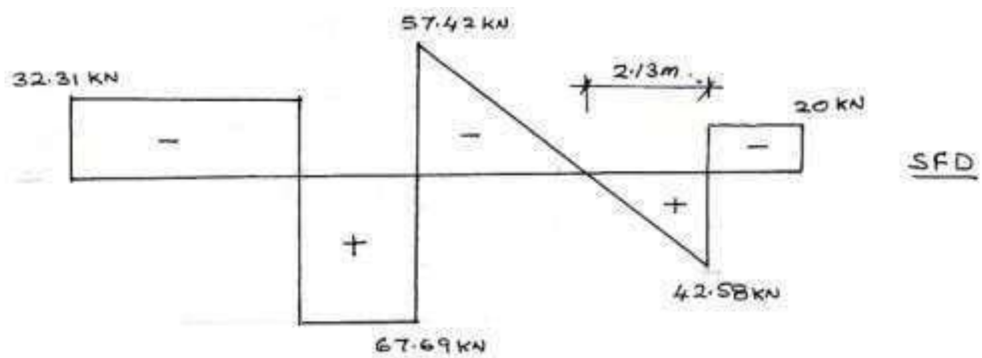
$M_{CD} = -30$ KNM



Reactions: Consider free body diagram of beam AB, BC and CD as shown







Span AB

$$R_B \times 6 = 100 \times 4 + 67.11 - 61$$

$$R_B = 67.69 \text{ KN}$$

$$R_A = 100 - R_B = 32.31 \text{ KN}$$

Span BC

$$R_C \times 5 = 20 \times \frac{5}{2} \times 5 + 30 - 67.11$$

$$R_C = 42.58 \text{ KN}$$

$$R_B = 20 \times 5 - R_B = 57.42 \text{ KN}$$

Maximum Bending Moments:

$$\text{Max} = 133.33 - 61 - \left(\frac{67.11 - 61}{6} \times 4\right) = 68.26 \text{ KN M}$$

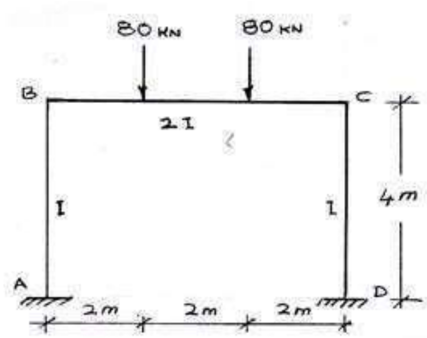
SpanBC: where SF=0, consider SF equation with C as reference

$$S_X = 42.58 - 20x = 0$$

$$x = \frac{42.58}{20} = 2.13 \text{ m}$$

$$\therefore M_{max} = 42.58 \times 2.13 - 20 \times \frac{2.13^2}{2} - 30 = 15.26 \text{ KNM}$$

5) Analyse the portal frame shown in figure and also drawn bending moment and shear force diagram



Solution:

FEM

$$F_{BC} = -\frac{W_1 ab^2}{L^2} - \frac{W_2 cd^2}{L^2} = -\frac{80 \times 2 \times 4^2}{6^2} - \frac{80 \times 4 \times 2^2}{6^2} = -106.67 \text{ KNM}$$

$$F_{CB} = +\frac{W_1 a^2 b}{L^2} + \frac{W_2 c^2 d}{L^2} = +106.67 \text{ KNM}$$

Slope deflection equations:

$$M_{AB} = F_{AB} + \frac{2EI}{L}(2\theta_A + \theta_B) = 0 + \frac{2EI}{4}(0 + \theta_B) = \frac{1}{2}EI\theta_B \quad \text{------------>} (1)$$

$$M_{BA} = F_{BA} + \frac{2EI}{L}(2\theta_B + \theta_A) = 0 + \frac{2EI}{4}(2\theta_B + 0) = EI\theta_B \quad \text{---------->} (2)$$

$$M_{BC} = F_{BC} + \frac{2EI}{L}(2\theta_B + \theta_C)$$

$$= -106.67 + \frac{2E2I}{6}(2\theta_B + \theta_C) = -106.67 + \frac{4}{3}EI\theta_B + \frac{2}{3}EI\theta_C \quad \text{-------->} (3)$$

$$M_{CB} = F_{CB} + \frac{2EI}{L}(2\theta_C + \theta_B)$$

$$= +106.67 + \frac{2E2I}{6}(2\theta_C + \theta_B) = +106.67 + \frac{4}{3}EI\theta_C + \frac{2}{3}EI\theta_B \quad \text{------------->} (4)$$

$$M_{CD} = F_{CD} + \frac{2EI}{L}(2\theta_C + \theta_D)$$

$$= 0 + \frac{2EI}{4}(2\theta_C + 0) = EI\theta_C \quad \text{-->} (5)$$

$$M_{DC} = F_{DC} + \frac{2EI}{L}(2\theta_D + \theta_C)$$

$$= 0 + \frac{2EI}{4}(0 + \theta_C) = \frac{1}{2} EI\theta_C \quad \text{---------} > (6)$$

Now $M_{BA} + M_{BC} = -106.67 + \frac{7}{3} EI\theta_B + \frac{2}{3} EI\theta_C = 0$ ----------------> (7)

$$\left.\begin{aligned} -746.69 + \frac{49}{3} EI\theta_B + \frac{14}{3} EI\theta_C = 0 \\ +213.34 + \frac{4}{3} EI\theta_B + \frac{14}{3} EI\theta_C = 0 \end{aligned}\right\} \text{subtracts}$$

$$-960.03 + \frac{45}{3} EI\theta_B = 0$$

$$EI\theta_B = +960.03 \times \frac{3}{45} = +64 \quad \text{Clockwise}$$

Using equation (7)

$$EI\theta_C = -\frac{3}{2}\left[-106.67 + \frac{7}{3} EI\theta_B\right]$$

$$= -\frac{3}{2}\left[-106.67 + \frac{7}{3} \times 64\right] = -64 \quad \text{Anticlockwise}$$

∴ Final moments are

$$M_{AB} = +\frac{64}{2} = +32 \quad \text{KNM}$$

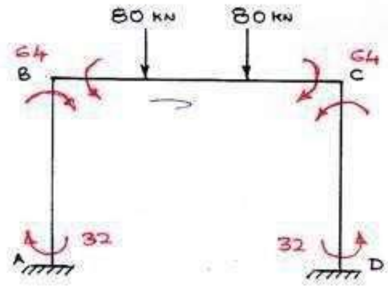
$$M_{BA} = 64 \quad \text{KNM}$$

$$M_{BC} = -106.67 + \frac{4}{3}64 + \frac{2}{3}(-64) = -64 \quad \text{KNM}$$

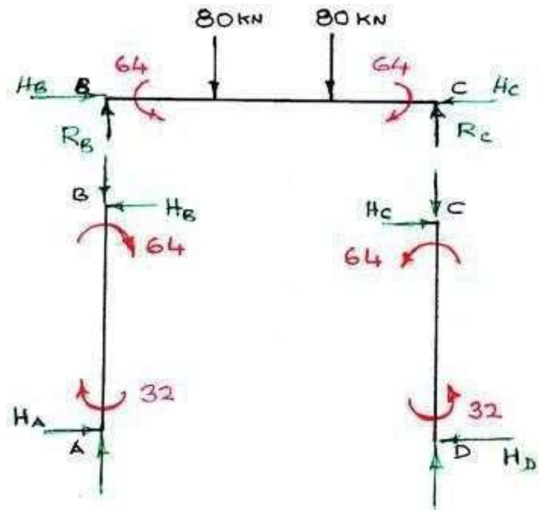
$$M_{CB} = +106.67 + \frac{4}{3}(-64) + \frac{2}{3}(64) = +64 \quad \text{KNM}$$

$$M_{CD} = -64 \quad \text{KNM}$$

$$M_{DC} = -\frac{1}{2} 64 = -32 \quad \text{KNM}$$



Consider free body diagrams of beam and columns as shown



By symmetrical we can write

$R_A = R_B = 60$ KNM

$R_D = R_C = 80$ KNM

$\sum M_B = 0$

$H_A \times 4 = 64 + 32$

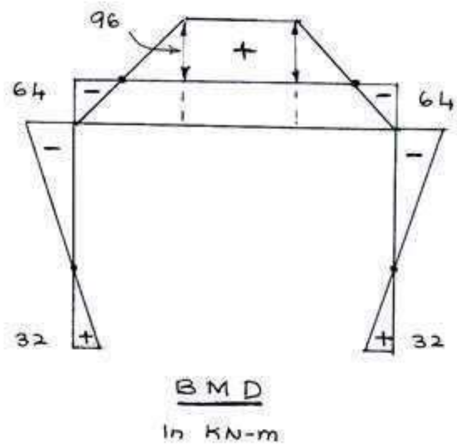
$\therefore\ H_A = 24$ KN

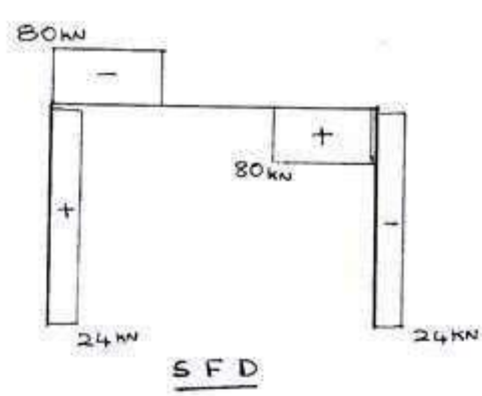
Apply

$\sum M_C = 0$

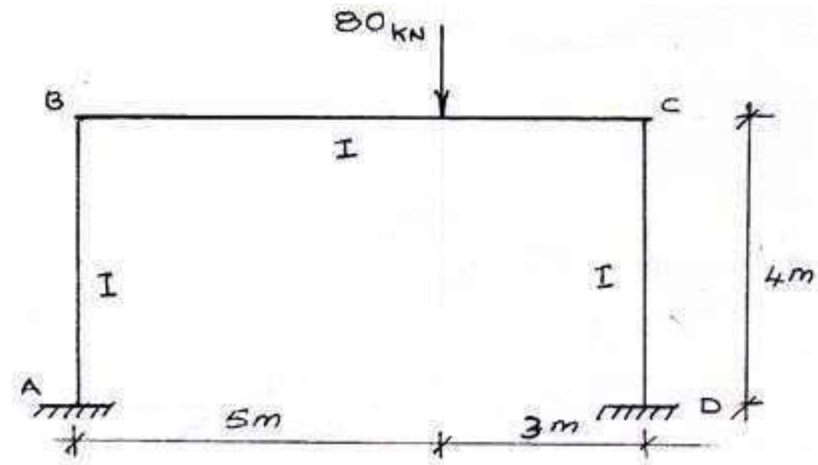
$H_A \times 4 = 64 + 32$

$\therefore\ H_D = 24$ KN





6)Analyse the portal frame and then draw the bending moment diagram



Solution:

Assume sway to right.

Here $\theta_A = 0, \theta_D = 0, \theta_B \neq 0, \theta_D = 0$

FEMS:

$$F_{BC} = -\frac{Wab^2}{L^2} = -\frac{80 \times 5 \times 3^2}{8^2} = -56.25 \quad KNM$$

$$F_{CB} = +\frac{Wa^2b}{L^2} = +\frac{80 \times 5^2 \times 3}{8^2} = +93.75 \quad KNM$$

Slope deflection equations

$$M_{AB} = F_{AB} + \frac{2EI}{L}\left(2\theta_A + \theta_B - \frac{3\delta}{L}\right)$$

$$= 0 + \frac{2EI}{4}\left(0 + \theta_B - \frac{3\delta}{4}\right) = \frac{1}{2}EI\theta_B - \frac{3}{8}EI\delta \qquad \text{----------->} (1)$$

$$M_{BA} = F_{BA} + \frac{2EI}{L}\left(2\theta_B + \theta_A - \frac{3\delta}{L}\right)$$

$$= 0 + \frac{2EI}{4}\left(2\theta_B + 0 - \frac{3\delta}{4}\right) = EI\theta_B - \frac{3}{8}EI\delta \qquad \text{----------->} (2)$$

$$M_{BC} = F_{BC} + \frac{2EI}{L}\left(2\theta_B + \theta_C\right)$$

$$= -56.25 + \frac{2EI}{8}\left(2\theta_B + \theta_C\right) = -56.25 + \frac{1}{2}EI\theta_B + \frac{1}{4}EI\theta_C \qquad ------------>(3)$$

$$M_{CB} = F_{CB} + \frac{2EI}{L}\left(2\theta_C + \theta_B\right)$$

$$= +93.75 + \frac{2EI}{8}\left(2\theta_C + \theta_B\right) = 93.75 + \frac{1}{2}EI\theta_C + \frac{1}{4}EI\theta_B \qquad ------------>(4)$$

$$M_{CD} = F_{CD} + \frac{2EI}{L}\left(2\theta_C + \theta_D - \frac{3\delta}{L}\right)$$

$$= 0 + \frac{2EI}{4}\left(2\theta_C + 0 - \frac{3\delta}{4}\right) = EI\theta_C - \frac{3}{8}EI\delta \qquad ------------>(5)$$

$$M_{DC} = F_{DC} + \frac{2EI}{L}\left(2\theta_D + \theta_C - \frac{3\delta}{L}\right)$$

$$= 0 + \frac{2EI}{4}\left(0 + \theta_C - \frac{3\delta}{4}\right) = \frac{1}{2}EI\theta_C - \frac{3}{8}EI\delta \qquad ------------>(6)$$

$M_{BA} + M_{BC} = 0$ --> Joint conditions

$M_{CB} + M_{CD} = 0$

$H_A + H_D + \sum P_H = 0$ ---> Shear condition

i.e, $\frac{M_{AB} + M_{BA}}{4} + \frac{M_{CD} + M_{DC}}{4} = 0$

$\therefore \quad M_{AB} + M_{BA} + M_{CD} + M_{DC} = 0$

Now, $M_{BA} + M_{BC} = EI\theta_B - \frac{3}{8}EI\delta - 56.25 + \frac{1}{2}EI\theta_B + \frac{1}{4}EI\theta_C = 0$

$$= -56.25 + \frac{3}{2}EI\theta_B + \frac{1}{4}EI\theta_C - \frac{3}{8}EI\delta = 0 \qquad ------->(7)$$

And, $M_{CB} + M_{CD} = 93.75 + \frac{1}{2}EI\theta_C + \frac{1}{4}EI\theta_B + EI\theta_C - \frac{3}{8}EI\delta = 0$

$$= 93.75 + \frac{1}{4}EI\theta_B + \frac{3}{2}EI\theta_C - \frac{3}{8}EI\delta = 0 \qquad ------->(8)$$

And, $M_{AB} + M_{BA} + M_{CD} + M_{DC} = \frac{1}{2}EI\theta_B - \frac{3}{8}EI\delta + EI\theta_B - \frac{3}{8}EI\delta + EI\theta_C - \frac{3}{8}EI\delta + \frac{1}{2}EI\theta_C - \frac{3}{8}EI\delta$

$$= \frac{3}{2}EI\theta_B + \frac{3}{2}EI\theta_C - \frac{3}{2}EI\delta = 0 \qquad ------->(9)$$

From (9) $EI\delta = EI\theta_B + EI\theta_C$

Substitute in (7) & (8)

Eqn(7)

$$-56.25 + \frac{3}{2}EI\theta_B + \frac{1}{4}EI\theta_C - \frac{3}{8}\left[EI\theta_B + EI\theta_C\right] = 0$$

$$-56.25 + \frac{9}{8}EI\theta_B - \frac{1}{8}EI\theta_C = 0 \quad \text{-----------} > (10)$$

Eqn(8)

$$+93.75 + \frac{1}{4}EI\theta_B + \frac{3}{2}EI\theta_C - \frac{3}{8}\left[EI\theta_B + EI\theta_C\right] = 0$$

$$+93.75 - \frac{1}{8}EI\theta_B + \frac{9}{8}EI\theta_C = 0 \text{---} > (11)$$

Solving equations (10) & (11) we get $EI\theta_B = 41.25$

By Equation (10)

$$EI\theta_C = 8\left[-56.25 + \frac{9}{8}EI\theta_B\right]$$

$$= 8\left[-56.25 + \frac{9}{8}41.25\right] = -78.75$$

$$\therefore EI\delta = EI\theta_B + EI\theta_C = 41.25 - 78.75 = -37.5$$

Hence

$$EI\theta_B = 41.25,\ EI\theta_C = -78.75,\ EI\delta = -37.5$$

$$M_{AB} = \frac{1}{2}(41.25) - \frac{3}{8}(-37.5) = +34.69 \text{ KNM}$$

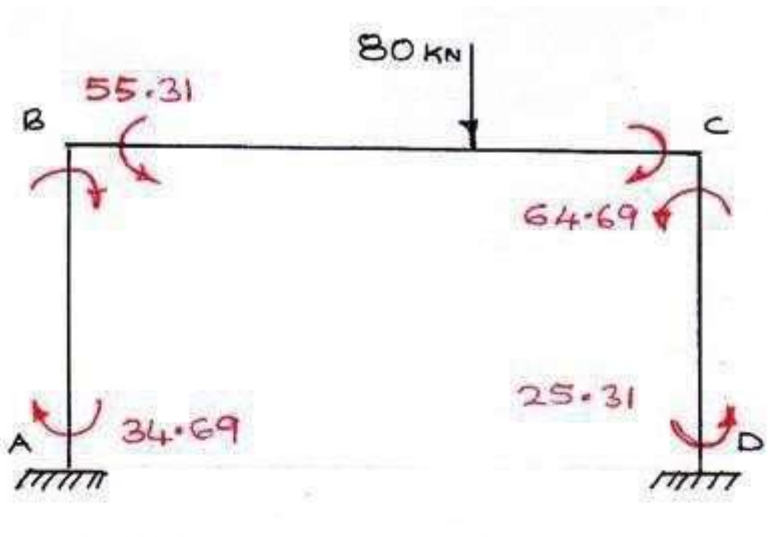
$$M_{BA} = 41.25 - \frac{3}{8}(-37.5) = +55.31 \text{KNM}$$

$$M_{BC} = -56.25 + \frac{1}{2}(41.25) + \frac{1}{4}(-78.75) = -55.31 \text{KNM}$$

$$M_{CB} = 93.75 + \frac{1}{2}(-78.75) + \frac{1}{4}(41.75) = +64.69 \text{ KNM}$$

$$M_{CD} = -78.75 - \frac{3}{8}(-37.5) = -64.69 \text{ KNM}$$

$$M_{DC} = \frac{1}{2}(-78.75) - \frac{3}{8}(-37.5) = -25.31 \text{KNM}$$



Reactions

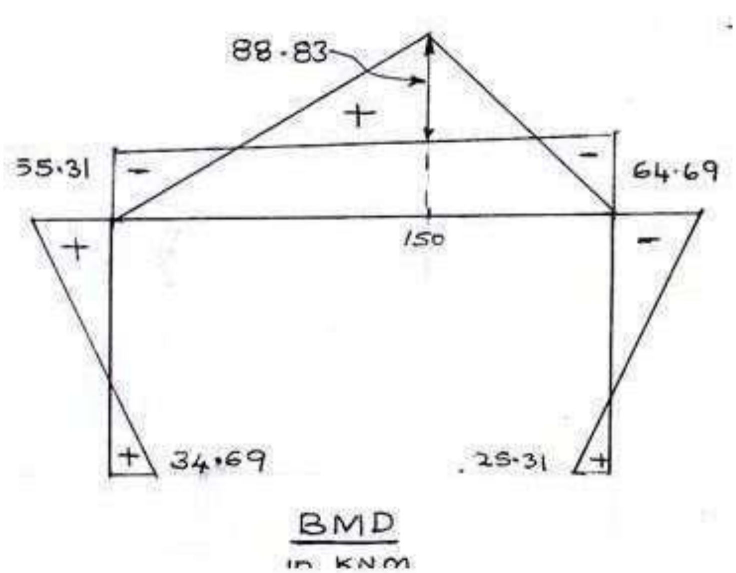
SpanBC:

$$R_B = \frac{55.31 - 64.69 + 80 \times 3}{8} = 28.83 \text{ KN}$$

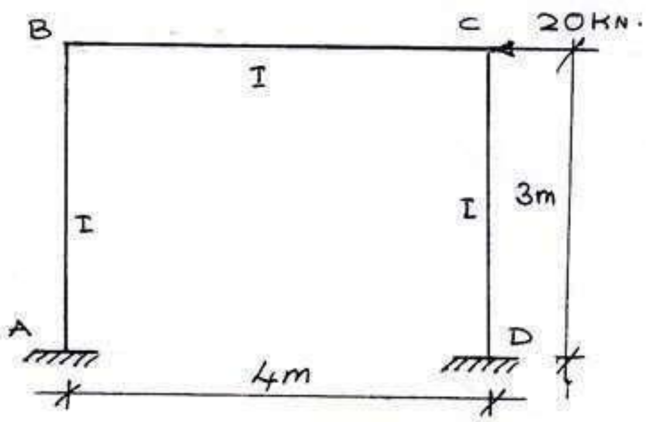
$$\therefore \; R_C = 80 - R_B = 51.17$$

Column CD:

$$H_D = \frac{64.69 + 25.31}{4} = 22.5$$



7) Frame ABCD is subjected to a horizontal force of 20 KN at joint C as shown in figure. Analyse and draw bending moment diagram.



Solution:

Frame is Symmetrical and unsymmetrical loaded hence there is a sway. Assume sway towards right

FEM

$$F_{AB} = F_{BA} = F_{BC} = F_{CB} = F_{CD} = F_{DC} = 0$$

Slope deflection equations are

$$M_{AB} = F_{AB} + \frac{2EI}{L}\left(2\theta_A + \theta_B - \frac{3\delta}{L}\right)$$

$$= \frac{2EI}{3}\left(\theta_B - \frac{3\delta}{3}\right)$$

$$= \frac{2}{3}EI\theta_B - \frac{2}{3}EI\delta \qquad \text{------------} > (1)$$

$$M_{BA} = F_{BA} + \frac{2EI}{L}\left(2\theta_B + \theta_A - \frac{3\delta}{L}\right)$$

$$= \frac{2EI}{3}\left(2\theta_B - \frac{3\delta}{3}\right)$$

$$= \frac{4}{3}EI\theta_B - \frac{2}{3}EI\delta \qquad \text{------} > (2)$$

$$M_{BC} = F_{BC} + \frac{2EI}{L}(2\theta_B + \theta_C)$$

$$= \frac{2EI}{4}(2\theta_B + \theta_C)$$

$$= EI\theta_B + 0.5\,EI\theta_C \text{ -------------------> } (3)$$

$$M_{CB} = F_{CB} + \frac{2EI}{L}(2\theta_C + \theta_B)$$

$$= \frac{2EI}{4}(2\theta_C + \theta_B)$$

$$= EI\theta_C + 0.5\,EI\theta_B \text{ ------------> } (4)$$

$$M_{CD} = F_{CD} + \frac{2EI}{L}\left(2\theta_C + \theta_D - \frac{3\delta}{L}\right)$$

$$= \frac{2EI}{3}\left[2\theta_C - \frac{3\delta}{3}\right]$$

$$= \frac{4}{3}EI\theta_C - \frac{2}{3}EI\delta \text{ ------> } (5)$$

$$M_{DC} = F_{DC} + \frac{2EI}{L}\left[2\theta_D + \theta_C - \frac{3\delta}{L}\right]$$

$$= \frac{2\,EI}{3}\left[\theta_c - \frac{3\delta}{3}\right]$$

$$= \frac{2}{3}EI\theta_C - \frac{2}{3}EI\delta \text{ -------------> } (6)$$

I. $M_{BA} + M_{BC} = 0$

II. $M_{CB} + M_{CD} = 0$

III. $H_A + H_D - 20 = 0$

i.e, $\frac{M_{AB} + M_{BA}}{3} + \frac{M_{CD} + M_{DC}}{3} - 20 = 0$

$$M_{AB} + M_{BA} + M_{CD} + M_{DC} - 60 = 0$$

Now $M_{BA} + M_{BC} = \frac{4}{3}EI\theta_B - \frac{2}{3}EI\delta + EI\theta_B + 0.5EI\theta_C$

$$= \frac{7}{3}EI\theta_B + 0.5EI\theta_C - \frac{2}{3}EI\delta = 0 \text{ ------> } (7)$$

and $M_{CB} + M_{CD} = EI\theta_C + 0.5EI\theta_B + \frac{4}{3}EI\theta_C - \frac{2}{3}EI\delta$

$$= 0.5EI\theta_B + \frac{7}{3}EI\theta_C - \frac{2}{3}EI\delta = 0 \text{ ------> } (8)$$

$$\text{and } M_{AB} + M_{BA} + M_{CD} + M_{DC} - 60 = \frac{2}{3}EI\theta_B - \frac{2}{3}EI\delta + \frac{4}{3}EI\theta_B - \frac{2}{3}EI\delta + \frac{4}{3}EI\theta_C$$

$$- \frac{2}{3}EI\delta + \frac{2}{3}EI\theta_C - \frac{2}{3}EI\delta - 60$$

$$= 2EI\theta_B + 2EI\theta_C - \frac{8}{3}EI\delta - 60 = 0 \text{ --------- } (9)$$

$EI\theta_B = -8.18,$

$EI\theta_C = -8.18,$

$EI\delta = -34.77$

$$M_{AB} = \frac{2}{3}(-8.18) - \frac{2}{3}(-34.77) = +17.73 \text{ KNM}$$

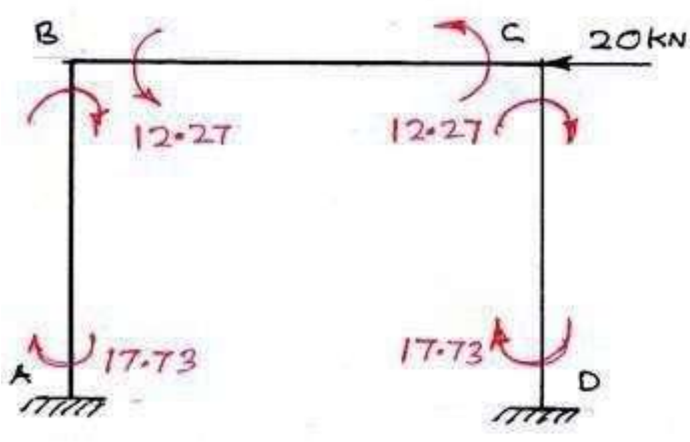
$$M_{BA} = \frac{4}{3}(-8.18) - \frac{2}{3}(-34.77) = +12.27 \text{ KNM}$$

$$M_{BC} = 0 - 8.18 + 0.5\,(-8.18) = -12.27 \text{ KNM}$$

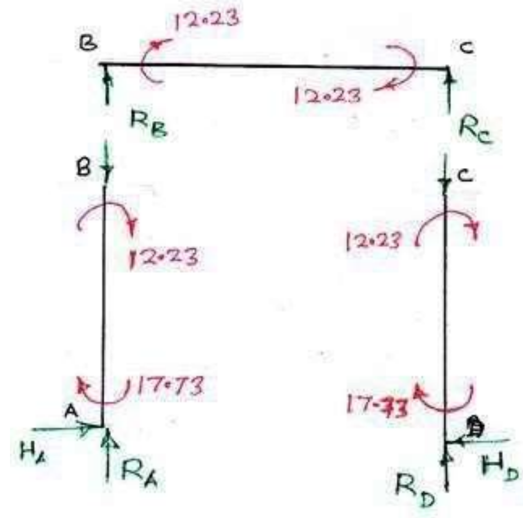
$$M_{CB} = 0.5\,(-8.18) - 8.18 = -12.27 \text{ KNM}$$

$$M_{CD} = \frac{4}{3}(-8.18) - \frac{2}{3}(-34.77) = +12.27 \text{ KNM}$$

$$M_{DC} = \frac{2}{3}(-8.18) - \frac{2}{3}(-34.77) = +17.73 \text{ KNM}$$



Reactions: Consider the free body diagram of the members



Member AB:

$$H_A = \frac{17.73 + 12.27}{3} = 10 \text{ KN}$$

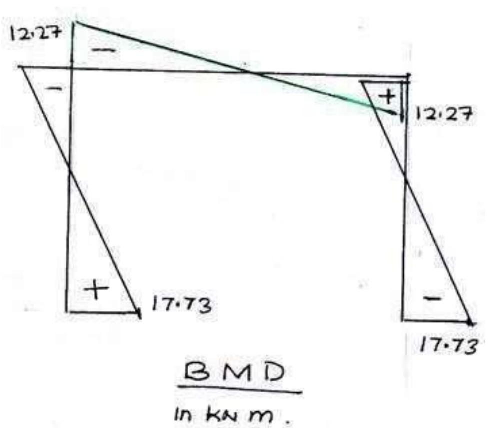
MemberBC:

$$R_C = \frac{12.27 + 12.27}{4} = 6.135 \text{ KN}$$

$\therefore R_B = -R_C = -6.135$ KN -ve sign indicates direction of R_B downwards

Member CD:

$H_D = \frac{-17.73 - 12.27}{3} = -10$ KN -ve sign indicates the direction of H_D is left to right



UNIT-V

MOMENT DISTRIBUTION METHOD

Distribution and carryover of moments – Stiffness and carry over factors – Analysis of continuous beams – Plane rigid frames with and without sway – Neylor"s simplification.

Hardy Cross (1885-1959)

- Moment Distribution is an **iterative method** of solving an **indeterminate Structure**.
- Moment distribution method was first introduced by **Hardy Cross in 1932.**
- Moment distribution is ***suitable for analysis of all types of indeterminate beams and rigid frames***.
- It is also called a **'relaxation method'** and it consists of successive ***approximations using a series of cycles***, ***each converging towards final result***.
- It is comparatively ***easier than slope deflection method***. It involves solving number of simultaneous equations with several unknowns, but in this method does not involve any simultaneous equations.
- It is **very easily** remembered and **extremely useful for checking computer output** of **highly indeterminate structures**.
- It is ***widely used*** in the analysis of ***all types of indeterminate beams and rigid frames***.
- The moment-distribution method was very popular among engineers.
- It is ***very simple*** and is being used even today for ***preliminary analysis of small structures***.
- The ***primary concept*** used in this methods are,
  - Fixed End Moments
  - Relative **or** Beam Stiffness **or** Stiffness factor
  - Distribution factor
  - Carry over moment or Carry over factor

Basic Concepts

- In moment-distribution method, counterclockwise beam end moments are taken as positive.
- The counterclockwise beam end moments produce clockwise moments on the joint.
- ***Note the sign convention:***

Anti-clockwise is positive (+)

Clockwise is negative (-)

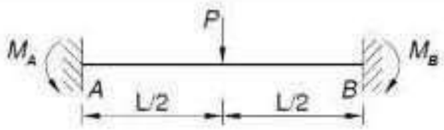
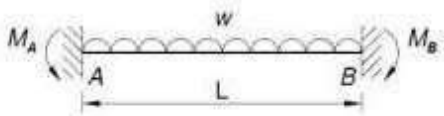
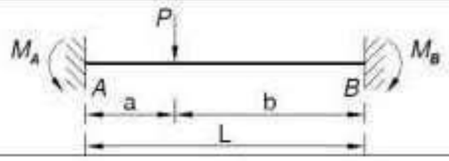
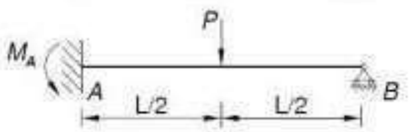
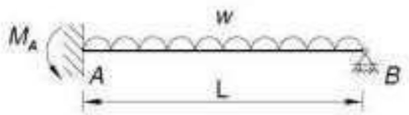
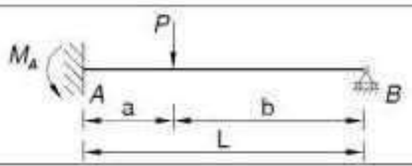
Assumptions in moment distribution method

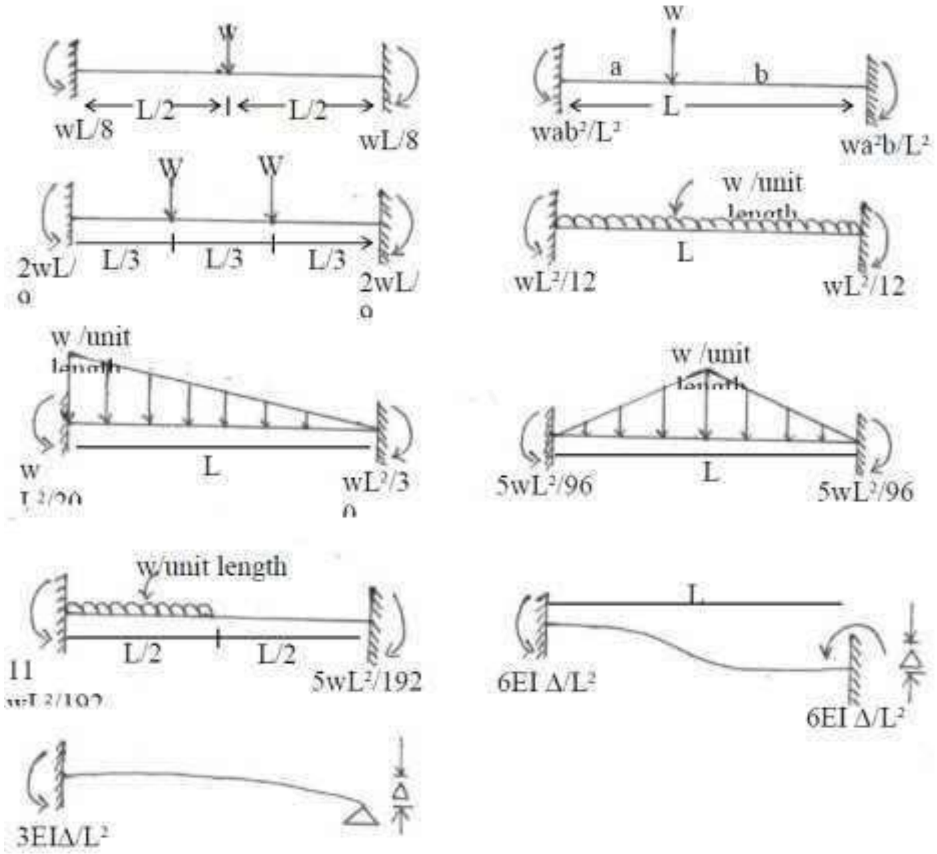
- ***All the members*** of the structures are ***assumed to be fixed*** and fixed end moments due to external loads are obtained.
- All the hinged joints are released by applying an equal and opposite moment.
- The joints are allowed to deflect (rotate) one after the other by releasing them successively.
- The unbalanced moment at the joint is shared by the members connected at the joint when it is released.
- The unbalanced moment at a joint is distributed in to the two spans with their distribution factor.

- Hardy cross method makes use of the ability of various structural members at a joint to sustain moments in proportional to their relative stiffness.

Fixed End Moments

- All members of a given frame are initially assumed fixed at both ends.
- The loads acting on these fixed beams produce fixed end moments at the ends.
- FEM are the moments exerted by the supports on the beam ends.
- These (non-existent) moments keep the rotations at the ends of each member zero.

M_A	Configuration	M_B
$+\frac{PL}{8}$		$-\frac{PL}{8}$
$+\frac{wL^2}{12}$		$-\frac{wL^2}{12}$
$+\frac{Pab^2}{L^2}$		$-\frac{Pa^2b}{L^2}$
$+\frac{3PL}{16}$		-
$+\frac{wL^2}{8}$		-
$+\frac{Pab(2L-a)}{2L^2}$		-



Relative or Beam Stiffness or Stiffness factor

- When a structural member of uniform section is subjected to a moment at one end, then the moment required so as to rotate that end to produce unit slope is called the ***stiffness of the member***.
- Stiffness is the member of ***force required to produce unit deflection***.
- It is also the moment required to produce unit rotation at a specified joint in a beam or a structure. It can be extended to denote the torque needed to produce unit twist.
- It is the moment required to rotate the end while acting on it through a unit rotation, without translation of the far end being
  - ✓ Beam is hinged or simply supported at both ends

    $k = 3\,EI / L$
  - ✓ Beam is hinged or simply supported at one end and fixed at other end

    $k = 4\,EI / L$
  - ✓ Stiffness of members in continuous beams and rigid frames
    - ***Stiffness of all intermediate members*** $k = 4\,EI / L$
    - ***Stiffness of edge members***,
      - If edge support is fixed $k = 4\,EI / L$
      - If edge support is hinged or roller $k = 3\,EI / L$
- Where, E = Young"s modulus of the beam material

  I = Moment of inertia of the beam

  L = Beam"s span length

Distribution factor

- When several members meet at a joint and a moment is applied at the joint to produce rotation without translation of the members, the moment is distributed among all the members meeting at that joint proportionate to their stiffness.
- Distribution factor = Relative stiffness / Sum of relative stiffness at the joint
- If there is 3 members,

  Distribution factors = $k_1 / (k_1+k_2+k_3)$, $k_2/ (k_1+k_2+k_3)$, $k_3/ (k_1+k_2+k_3)$

Carry over moment

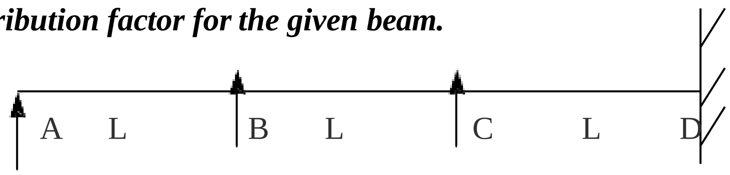
- Carry over moment: It is defined as the moment induced at the fixed end of the beam by the action of a moment applied at the other end, which is hinged.
- Carry over moment is the same nature of the applied moment.

Carry over factor (C.O):

- A moment applied at the hinged end B "carries over" to the fixed end „A‟, a moment equal to half the amount of applied moment and of the same rotational sense. C.O =0.5

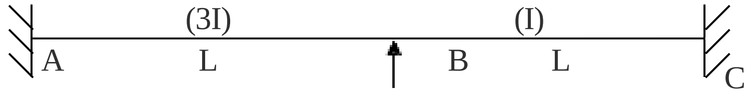
Problem:

1. Find the distribution factor for the given beam.



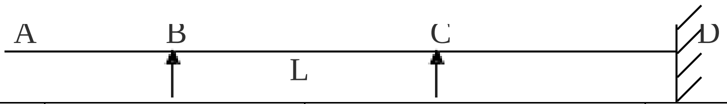
Joint	Member	Relative stiffness	Sum of Relative stiffness	Distribution factor
A	AB	4EI / L	4EI / L	(4EI / L) / (4EI / L) = 1
B	BA BC	3EI /L 4EI / L	3EI /L + 4EI / L = 7EI / L	(3EI / L) / (7EI / L)= 3/7 (4EI / L) / (7EI / L) = 4/7
C	CB CD	4EI / L 4EI / L	4EI / L + 4EI / L =8EI / L	(4EI / L) / (8EI / L) =4/8 (4EI / L) / (8EI / L)= 4/8
D	DC	4EI / L	4EI / L	(4EI / L)/ (4EI / L) = 1

2. Find the distribution factor for the given beam.



Joint	Member	Relative stiffness	Sum of Relative stiffness	Distribution factor
A	AB	4E (3I) / L	12EI / L	(12EI / L) / (12EI / L) = 1
B	BA BC	4E(3I) /L 4EI / L	12EI /L + 4EI / L = 16EI / L	(12EI / L) / (16EI / L)= 3/4 (4EI / L) / (16EI / L) = 1/4
C	CB	4EI / L	4EI / L	(4EI / L) / (4EI / L) =1

3. Find the distribution factor for the given beam.



Joint	Member	Relative stiffness	Sum of Relative	Distribution factor
B	BA BC	0 (no support) 3EI / L	3EI / L	0 (3EI / L) / (3EI / L) =1
C	CB CD	3EI / L 4EI / L	3EI /L + 4EI / L = 7EI / L	(3EI / L) / (7EI / L)= 3 / 7 (4EI / L) / (7EI / L) = 4 / 7
D	DC	4EI / L	4EI / L	(4EI / L) / (4EI / L) =1

Flexural Rigidity of Beams:

- The product of young"s modulus (E) and moment of inertia (I) is called Flexural Rigidity (EI) of Beams. The unit is N.mm^2.

Constant strength beam:

- If the flexural Rigidity (EI) is constant over the uniform section, it is called Constant strength beam.

Sway:

- Sway is the lateral movement of joints in a portal frame due to the unsymmetrical dimensions, loads, moments of inertia, end conditions, etc.

What are the situations where in sway will occur in portal frames?

- Eccentric or unsymmetrical loading
- Unsymmetrical geometry
- Different end conditions of the columns
- Non-uniform section of the members
- Unsymmetrical settlement of supports
- A combination of the above

What are symmetric and antisymmetric quantities in structural behaviour?

- When a symmetrical structure is loaded with symmetrical loading, the bending moment and deflected shape will be symmetrical about the same axis.
- Bending moment and deflection are symmetrical quantities

Steps involved in Moment Distribution Method:

1. Calculate fixed end moments due to applied loads following the same sign convention and procedure, which was adopted in the slope-deflection method.
2. Calculate relative stiffness.
3. Determine the distribution factors for various members framing into a particular joint.

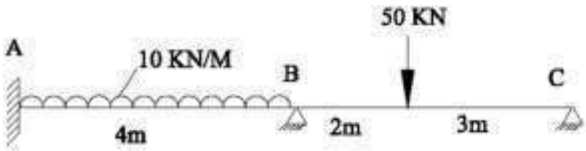
4. Distribute the net fixed end moments at the joints to various members by multiplying the net moment by their respective distribution factors in the first cycle.
5. In the second and subsequent cycles, carry-over moments from the far ends of the same member (carry-over moment will be half of the distributed moment).
6. Consider this carry-over moment as a fixed end moment and determine the balancing moment. This procedure is repeated from second cycle onwards till convergence

Advantages of Fixed Ends or Fixed Supports

1. Slope at the ends is zero.
2. Fixed beams are stiffer, stronger and more stable than SSB.
3. In case of fixed beams, fixed end moments will reduce the BM in each section.
4. The maximum defection is reduced.

Problem:

1. Analyse the frame given in figure by moment distribution method and draw the B.M.D & S.F.D



Step: 1 - Fixed end moment

M^F_{AB} = $-WL^2/12$ = $-10\times4^2/12$ = -13.33 KNM

M^F_{BA} = $WL^2/12$ = $10\times4^2/12$ = -13.33 KNM

M^F_{BC} = $-Wab^2/L^2$ = $-50\times2\times3^2/5^2$ = -36 KNM

M^F_{CB} = Wa^2b/L^2 = $50\times2^2\times3/5^2$ = 24 KNM

Step: 2 - Stiffness

K_{AB} = K_{BA} = $4EI/L$ = EI

K_{BC} = K_{CB} = $3EI/L$ = 0.6EI

Step: 3 - Distribution factor

Joint B

D^F_{BA} = $K_{BA}/(K_{BA}+K_{BC})$ = 0.63

D^F_{BC} = $K_{BC}/(K_{BA}+K_{BC})$ = 0.37

Step: 4 - Moment distribution

MEMBER	AB	B		CB
		BA	BC	
DF	0	0.67	0.33	0
FEM	-13.33	+13.33	-36	+24
BALANCING	0	0	0	-24
CF	0	0	-12	0
M	-13.33	+13.33	-48	0
BALANCING	0	21.84	12.83	0
CF	10.92	0	0	0
M-FINAL	**-2.4**	**35.17**	**-35.17**	**0**

Step: 5 - Reactions

Span AB:

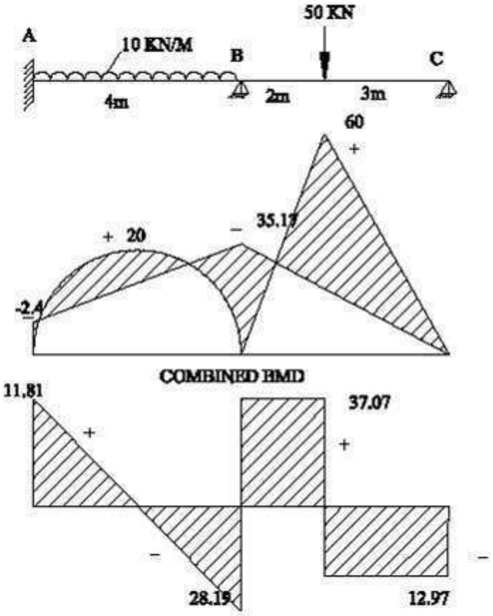
R_A = 11.81 KN

R_{B1} = 28.19 KN

Span BC:

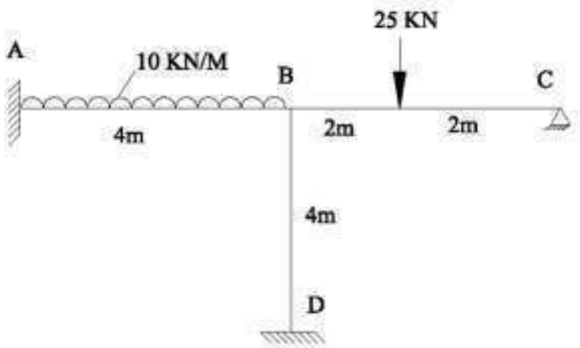
R_{B2} = 37.03 KN

R_C = 12.97 KN



SFD

2. Analyse the frame given in figure by moment distribution method and draw the B.M.D&S.F.D



Step: 1 - Fixed end moment

M^F_{AB} = $-WL^2/12$ = $-10\times4^2/12$ = -13.33 KNM

M^F_{BA} = $WL^2/12$ = $10\times4^2/12$ = -13.33 KNM

M^F_{BC} = $-WL/8$ = $-25\times4/8$ = -12.5 KNM

M^F_{CB} = $WL/8$ = $25\times4/8$ = 12.5 KNM

M^F_{BD} = 0

M^F_{DB} = 0

Step: 2 - Stiffness

K_{AB} = K_{BA} = $4EI/L$ = EI

K_{BC} = K_{CB} = $3EI/L$ = 0.75EI

K_{BD} = K_{DB} = $4EI/L$ = EI

Step: 3 - Distribution factor

Joint B

D^F_{BA} = $K_{BA}/(K_{BA}+K_{BC}+K_{BD})$ = 0.36

D^F_{BC} = $K_{BC}/(K_{BA}+K_{BC}+K_{BD})$ = 0.28

D^F_{BD} = $K_{BD}/(K_{BA}+K_{BC}+K_{BD})$ = 0.36

Step: 4 - Moment distribution

MEMBER	AB	B			DB	CB
		BA	**BC**	**BD**		
DF	0	0.36	0.28	0.36	0	
FEM	-13.33	+13.33	-12.5	0	0	+12.5
CF	0	0	-6.25	0	0	-12.5
M(initial)	-13.33	+13.33	-18.75	0	0	0
BALANCING	0	+1.95	1.52	1.95	0	0
MF	0.98	0	0	0	0.98	0
M-FINAL	**-12.35**	**15.28**	**-17.23**	**1.95**	**0.98**	**0**

Step: 5 - Find reactions:

Span AB:

R_A = 19.27 KN

R_{B1} = 20.73 KN

Span BC:

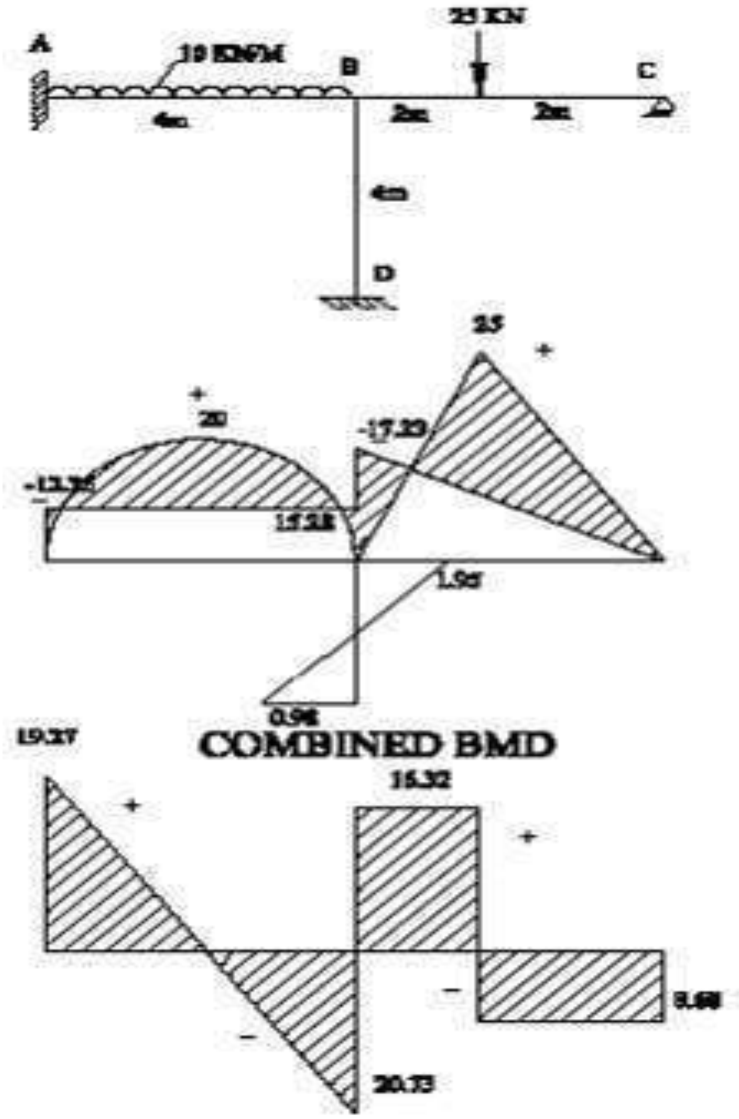
R_{B2} = 16.32 KN

R_C = 8.68 KN

Span BD:

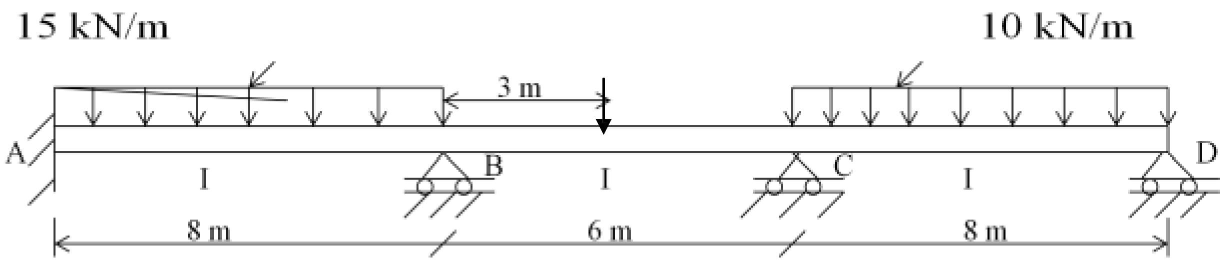
R_{B3} = -0.73 KN

R_D = 0.73 KN



SFD

3. The continuous beam ABCD, subjected to the given loads, as shown in Figure below. Assume that only rotation of joints occurs at B, C and D, and that no support displacements occur at B, C and D. Due to the applied loads in spans AB, BC and CD, rotations occur at B, C and D using moment distribution method.



Step: 1 - Fixed end moments

$$M_{AB} = -M_{BA} = -\frac{wl^2}{12} = -\frac{(15)(8)^2}{12} = -80\ kN.m$$

$$M_{BC} = -M_{CB} = -\frac{wl}{8} = -\frac{(150)(6)}{8} = -112.5\ kN.m$$

$$M_{CD} = -M_{DC} = -\frac{wl^2}{12} = -\frac{(10)(8)^2}{12} = -53.333\ kN.m$$

Step: 2 - Stiffness Factors (Unmodified Stiffness)

$$K_{AB} = K_{BA} = \frac{4EI}{L} = \frac{(4)(EI)}{8} = 0.5EI$$

$$K_{BC} = K_{CB} = \frac{4EI}{L} = \frac{(4)(EI)}{6} = 0.667EI$$

$$K_{CD} = \left[\frac{4EI}{8}\right] = \frac{4}{8}EI = 0.5EI$$

$$K_{DC} = \frac{4EI}{8} = 0.5EI$$

Step: 3 - Distribution Factors

$$DF_{AB} = \frac{K_{BA}}{K_{BA} + K_{wall}} = \frac{0.5EI}{0.5 + \infty\,(wall\ stiffness)} = 0.0$$

$$DF_{BA} = \frac{K_{BA}}{K_{BA} + K_{BC}} = \frac{0.5EI}{0.5EI + 0.667EI} = 0.4284$$

$$DF_{BC} = \frac{K_{BC}}{K_{BA} + K_{BC}} = \frac{0.667EI}{0.5EI + 0.667EI} = 0.5716$$

$$DF_{CB} = \frac{K_{CB}}{K_{CB} + K_{CD}} = \frac{0.667EI}{0.667EI + 0.500EI} = 0.5716$$

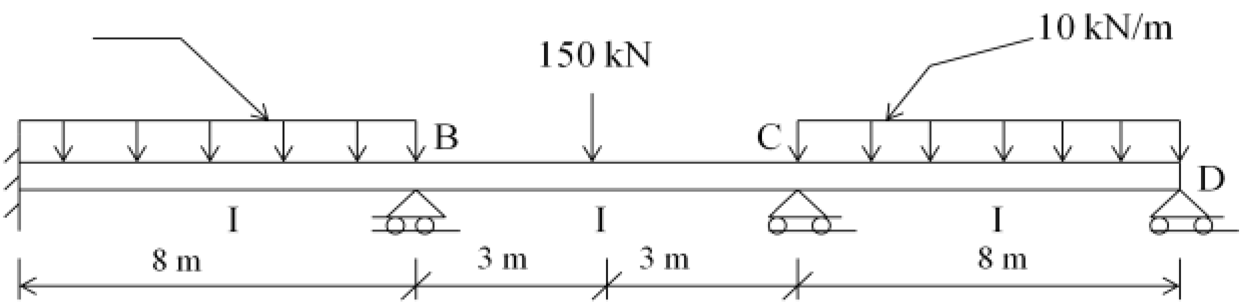
$$DF_{CD} = \frac{K_{CD}}{K_{CB} + K_{CD}} = \frac{0.500EI}{0.667EI + 0.500EI} = 0.4284$$

$$DF_{DC} = \frac{K_{DC}}{K_{DC}} = 1.00$$

Step: 4 - Moment Distribution

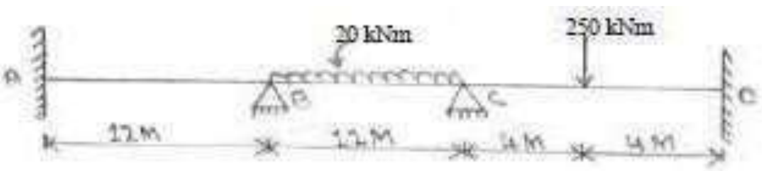
Joint	A	B		C		D
Member	**AB**	**BA**	**BC**	**CB**	**CD**	**DC**
DF	0	0.4284	0.5716	0.64	0.36	1
FEM	-80	80	-112.50	112.50	-53.33	53.33
I^{st} Distribution		13.923	18.577	-37.87	-21.3	-53.33
Carry over Moment	6.962		-18.93	9.289	-26.67	-10.65
2^{nd} Distribution		8.111	10.823	11.122	6.256	10.65
Carry over Moment	4.056		5.561	5.412	5.325	3.128
3^{rd} Distribution		-2.382	-3.179	-6.872	-3.865	-3.128
Carry over Moment	-1.191		-3.436	-1.59	-1.564	-1.933
4^{th} Distribution		1.472	1.964	2.019	1.135	1.933
Carry over Moment	0.736		1.01	0.982	0.967	0.568
5^{th} Distribution		-0.433	-0.577	-1.247	-0.702	-0.568
Carry over Moment						
M-FINAL	**-69.44**	**100.69**	**-100.7**	**-93.748**	**93.75**	**0**

Step: 5 - Computation of Shear Forces



Simply-supported reaction	60	60	75	75	40	40
End reaction due to left hand FEM	8.726	-8.726	16.665	-16.67	12.079	-12.08
End reaction due to right hand FEM	-12.5	12.498	-16.1	16.102	0	0
Summed-up moments	56.228	63.772	75.563	74.437	53.077	27.923

5. Analyse the beam as shown in figure by moment distribution method and draw the BMD. Assume EI is constant



Step: 1 - Fixed end moments

M^F_{AB} = 0

M^F_{BA} = 0

M^F_{BC} = $-WL^2/12$ = $-20\times12^2/12$ = -240 KNM

M^F_{CB} = $WL^2/12$ = $-20\times12^2/12$ = 240 KNM

M^F_{CD} = $-WL/8$ = $-250\times8/8$ = -250 KNM

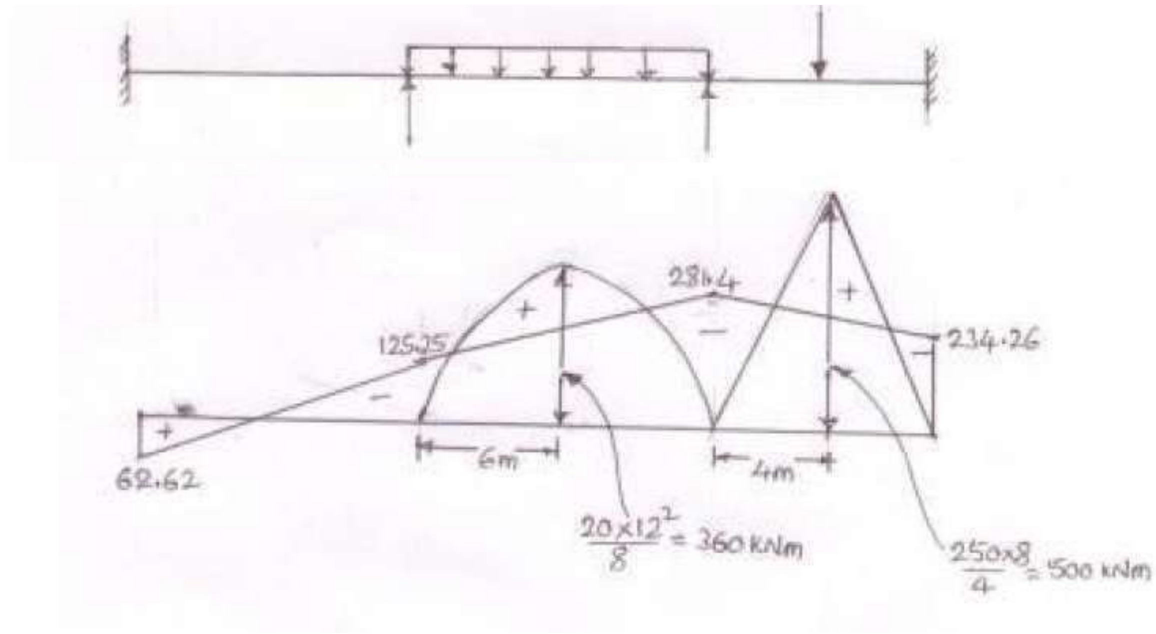
M^F_{DC} = $WL/8$ = $250\times8/8$ = 250 KNM

Step:2 - Distribution factor

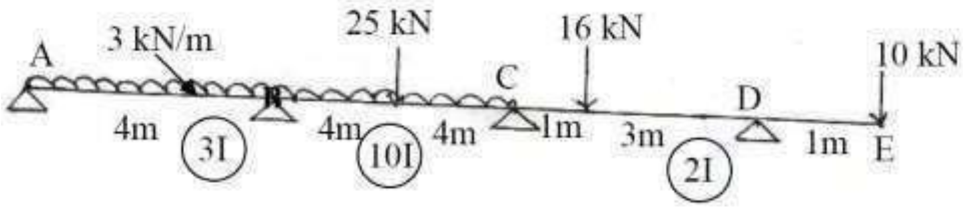
Joint	Member	Relative Stiffness (K)	ΣK	$D.F = (K / \Sigma K)$
B	BA	I / L = (I /12)	I / 6	0.50
	BC	I / L = (I /12)		0.50
C	CB	I / L = (I /12)	5I / 24	0.40
	CD	I / L = (I /8)		0.60

Step:3 – Moment Distribution

Jt	A	B		C		D
Member	AB	BA	BC	CB	CD	α
D.F	0	0.5	0.5	0.4	0.6	0
FEM	0	0	-240	+240	-250	+250
Balance		+120	+120	4	6	
C.O	60		2	60		3
Balance		-1	-1	-24	-36	
C.O	-0.5		-12	-0.5		-18
Balance		+6	+6	0.2	0.3	
C.O	3		0.1	3		0.15
Balance		-0.05	-0.05	-1.2	-1.8	
C.O	-0.03		-0.6	-0.03		-0.9
Balance		+0.3	+0.3	0.01	0.02	
C.O	0.15					0.01
Final moments	**62.62**	**125.25**	**-125.25**	**281.48**	**-281.48**	**234.26**



5. Analyze the continuous beam as shown in fig by moment distribution method and draw BMD & SFD



Step: 1 - Fixed end moments

FEM: $M_{FAB} = -\frac{3x4^2}{12} = -4\,kNm;\ M_{FBA} = 4\,kNm$

$M_{FBC} = -\frac{3x8^2}{12} - \frac{25\,x\,8}{8} = -41\ kNm \quad M_{FAB} = +\frac{3x8^2}{12} + \frac{25\,x\,8}{8} = +41\,kNm$

$M_{FDC} = \frac{16\,x\,1^2\,x\,3}{4^2} = +3\ kNm \qquad M_{DE} = -10\ x\ 1 = -10\,kNm$

$M_{FCD} = \frac{-16\times1\times3^2}{4^2} = -9\ kNm$

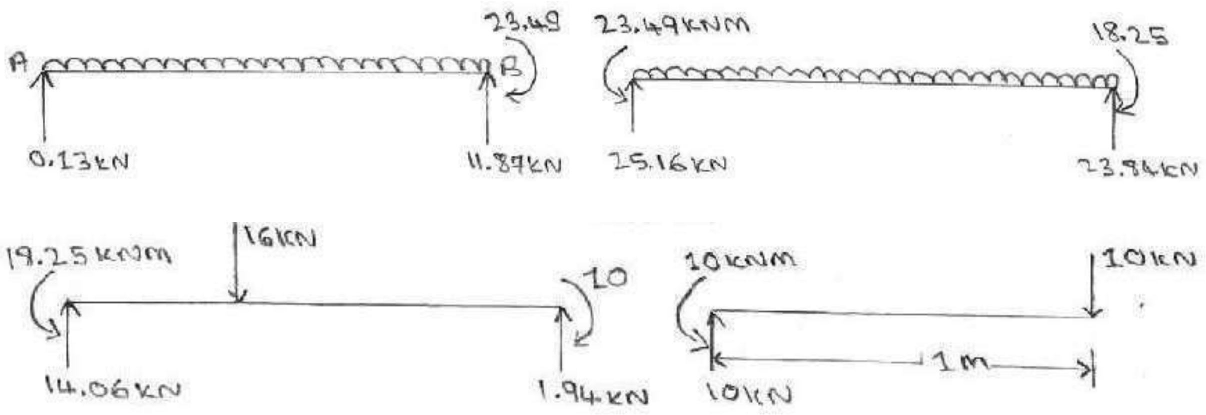
Step:2 - Distribution factor

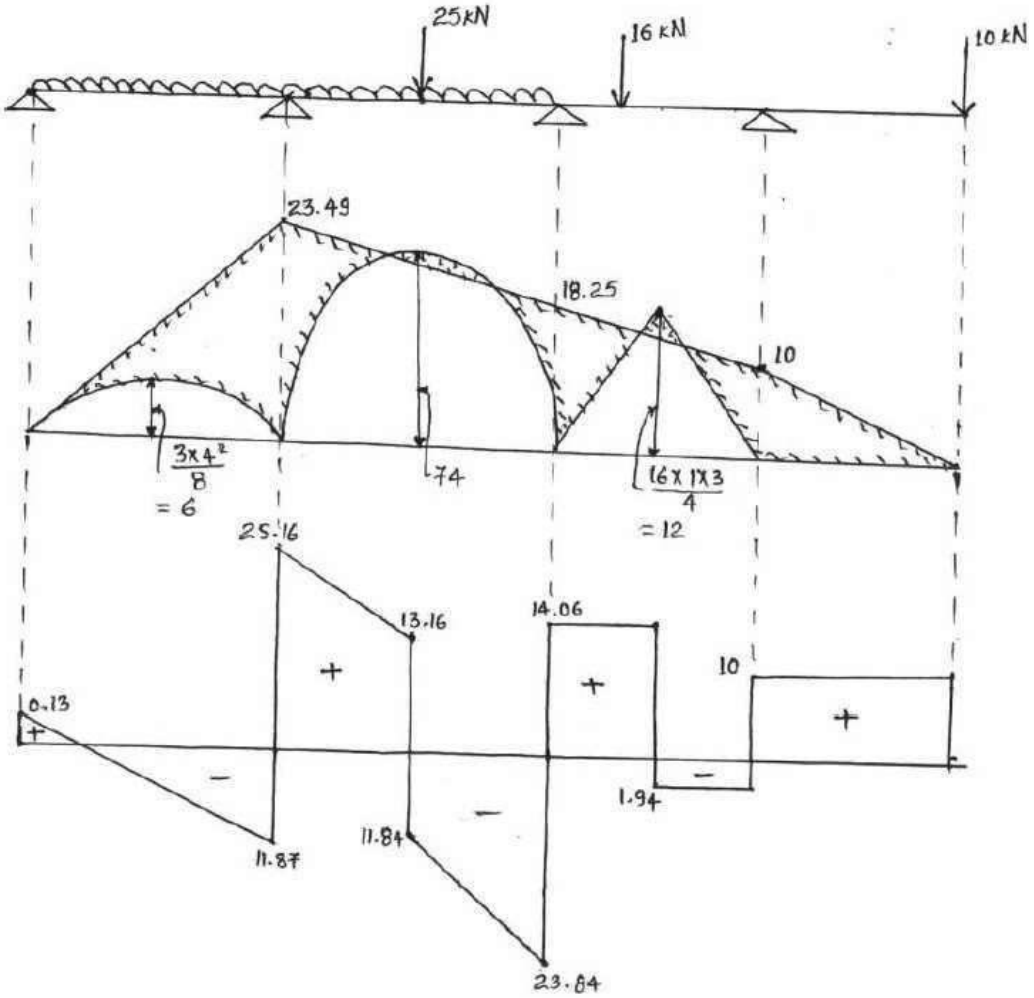
Jt.	Member	Relative stiffness (K)	ΣK	$DF = \frac{K}{\sum K}$
B	BA	$\frac{3}{4}x\frac{3I}{4} = 0.56I$	1.81I	0.31
	BC	10I/8 = 1.25I		0.69
C	CB	10I/8 = 1.25I	1.63I	0.77
	CD	$\frac{3}{4}x\frac{2I}{4} = 0.38I$		0.23

Step: 3 - Moment Distribution

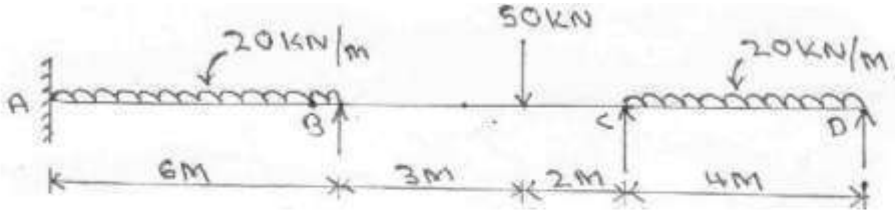
Jt	A	B		C		D	
Member	AB	BA	BC	CB	CD	DC	DE
D.F	1	0.31	0.69	0.77	0.23	1	01
FEM	-4	4	-41	+41	-9	3	-10
Release of joint A and adjusting moment at 'D'	+4	2			3.5	+7	
Initial moments	0	6	-41	41	-5.5	+10	-10
Balance		10.9	24.1	-27.3	-8.2		
C.O			-13.7	12.1			
Balance		4.2	9.5	-9.3	-2.8		
C.O			-4.7	4.8			
Balance		1.5	3.2	-3.7	-1.1		
C.O			-1.9	1.6			
Balance		0.6	1.3	-1.2	-0.4		
C.O			-0.6	0.7			
Balance		0.2	0.4	-0.5	-0.2		
C.O			-0.3	0.2			
Balance		0.09	0.21	-0.15	-0.05		
Final moments	**0**	**23.49**	**-23.49**	**18.25**	**-18.25**	**10**	**-10**

Step: 4 – BMD & SFD




25 kN
16 kN
10 kN
23.49
18.25
10
$\frac{3 \times 4^2}{8}$
= 6
74
$\frac{16 \times 1 \times 3}{4}$
= 12
25.16
13.16
14.06
10
0.13
+
−
11.87
11.84
23.84
1.94

6. Analyze the continues beam as shown in figure by moment distribution method and draw the B.M. diagrams



Support B sinks by 10mm, and take E = 2 x 10^5 N/mm², I = 1.2 x 10^{-4} m^4

<u>Step: 1 - Fixed end moments</u>

M_{FAB} = FEM due to load + FEM due to sinking

$$= \frac{-wl^2}{12} + \left[\frac{-6EI\Delta}{l^2}\right]$$

$$= \frac{-20 \, x \, 6^2}{12} - \frac{6 \, x \, 2 \, x \, 10^5 \, x \, 1.2 \, x \, 10^{-4} \, x \, 10^{12} \, x \, 10}{(6000)^2 \, x \, 10^6}$$

$= -60 - 40$

M_{FAB} = -100 kNm

M_{FBA} = FEM due to load + FEM due to sinking

$= +60 - 40$

M_{FBA} = +20 kNm

M_{FBC} = FEM due to loading + FEM due to sinking

$$= \frac{-Wab^2}{l^2} + \frac{6EI\Delta}{l^2}$$

$$= \frac{-50 \, x \, 3 \, x \, 2^2}{5^2} + \frac{6 \, x \, 2 \, x \, 10^5 \, 1.2 \, x \, 10^{-4} \, x \, 10^{12} \, x \, 10}{(5000)^2 \, x \, 10^6}$$

$= -24 + 57.6$

M_{FBA} = + 33.6 kNm

$$M_{FCB} = + \frac{Wa^2b}{l^2} + \frac{6EI\Delta}{l^2}$$

$$= \frac{50 \, x \, 3^2 \, x \, 2}{5^2} + 57.6$$

M_{FCB} = 93.6kNm

M_{FCD} = due to load only ($\because$ C & D are at some level)

$$M_{FCD} = \frac{-wl^2}{12} = \frac{-20 \, x \, 4^2}{12} = -26.67 \text{kNm}$$

M_{FDC} = + 26.67 kNm

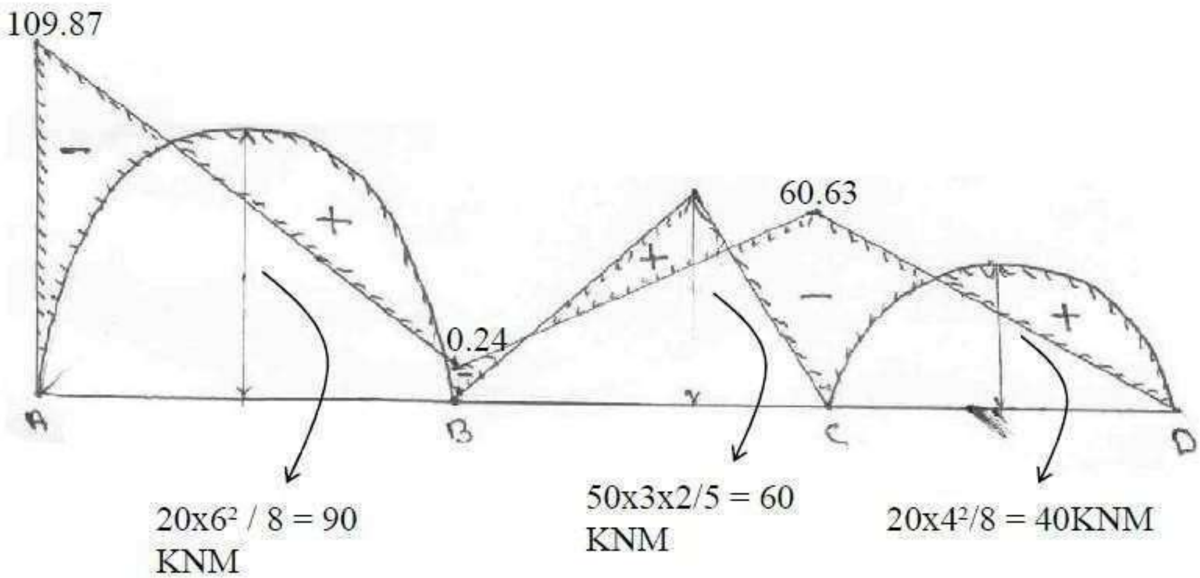
<u>Step:2 - Distribution factor</u>

Jt.	Member	Relative stiffness (K)	ΣK	DF = $\frac{K}{\sum K}$
B	BA BC	I/6 I/5	0.36I	0.46 0.54
C	CB CD	I/5 $\frac{3}{4} x \frac{I}{4} = 0.19I$	0.39I	0.51 0.49

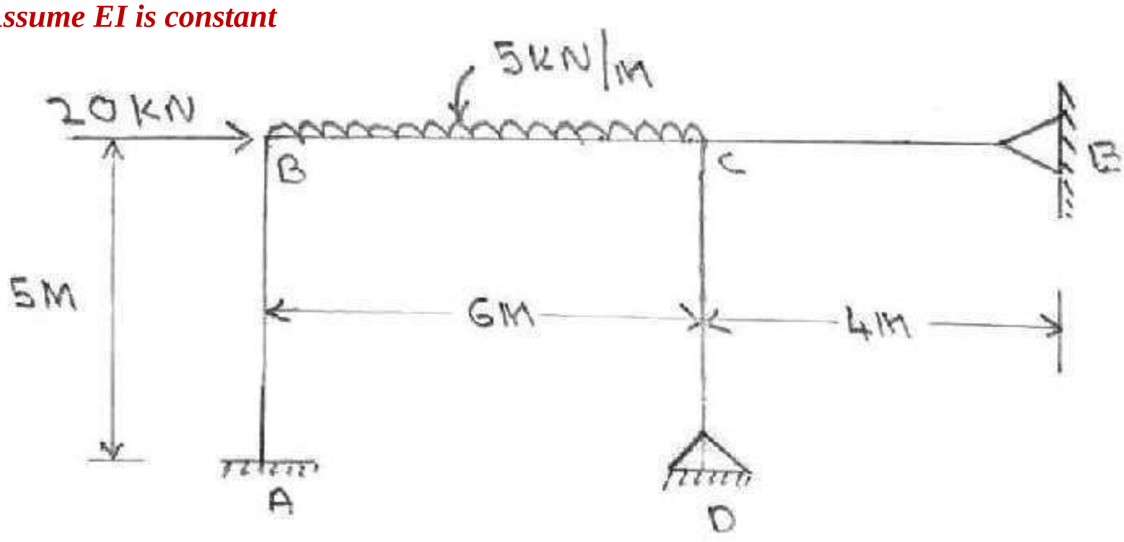
Step: 3 - Moment Distribution

Jt	A		B		C		D
Member	AB	BA	BC	CB	CD	DC	
D.F		0.46	0.54	0.51	0.49		
FEM	-100	+20	+33.6	+93.6	-26.67	+26.67	
Release jt. 'D'						-26.67	
CO					-13.34		
Initial moments	-100	+20	+33.6	+93.6	-40.01	0	
Balance		-24.66	-28.94	-27.33	-26.26		
C.O	-12.33		-13.67	--14.47			
Balance		+6.29	+7.38	+7.38	+7.09		
C.O	+3.15		+3.69	+3.69			
Balance		-1.7	-1.99	-1.88	-1.81		
C.O	-0.85		-0.94	-1			
Balance		+0.43	+0.51	+0.51	+0.49		
C.O	+0.22		+0.26	+0.26			
Balance		-0.12	-0.14	-0.13	-0.13		
C.O	-0.06						
Final moments	**-109.87**	**+0.24**	**-0.24**	**+60.63**	**-60.63**		

Step: 4 – BMD



6. Analysis the frame shown in figure by moment distribution method and draw BMD. Assume EI is constant



Step: 1 - Fixed end moments

$M_{FAB} = M_{FBA} = M_{FCD} = M_{FDC} = M_{FCE}$ $M_{FEC} = 0$

$M_{FBC} = -\dfrac{5 \times 6^2}{12} = -15\,\text{kNm}$

$M_{FCB} = +15$ kNm

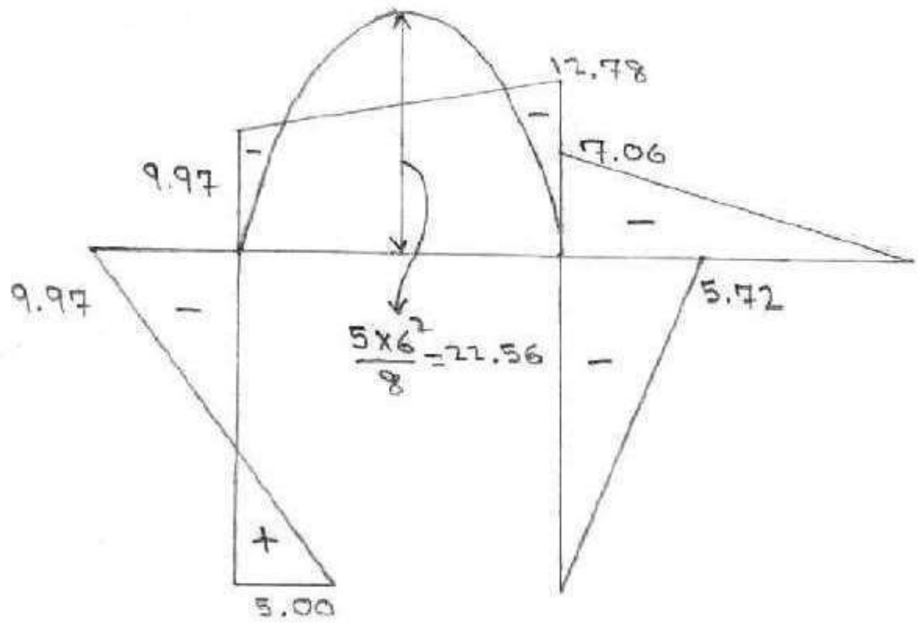
Step:2 - Distribution factor

Jt.	Member	Relative stiffness (K)	ΣK	DF = $\frac{K}{\sum K}$
B	BA BC	I/5 I/6	$\frac{11}{30}$I	0.55 0.45
C	CB CD CE	I/6 = 0.17 I $\frac{3}{4}$I/5 = 0.15 I $\frac{3}{4} \times \frac{I}{4} = 0.19$ I	0.51 I	0.33 0.3 0.37

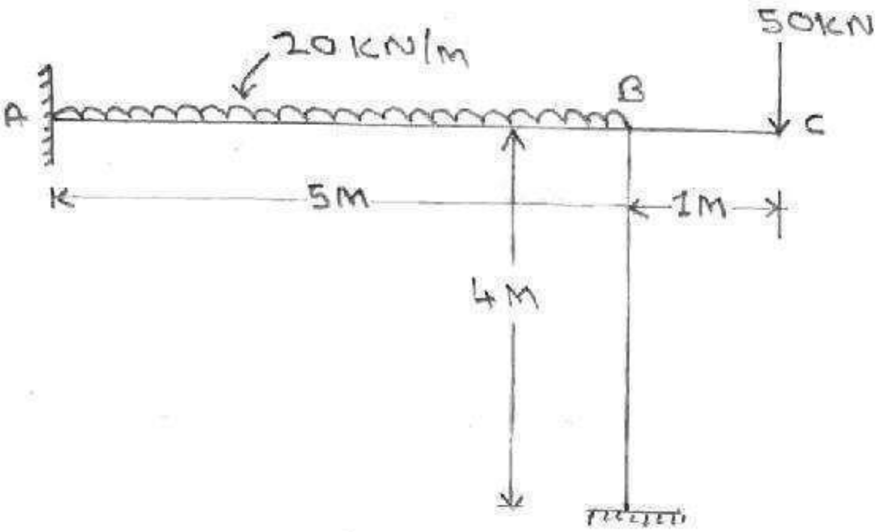
Step: 3 - Moment Distribution

Jt	A	B		C			D	E
Member	AB	BA	BC	CB	CD	CE	DC	EC
D.F	0	0.55	0.45	0.33	0.3	0.37	1	1
FEM	0	0	-15	+15	0	0	0	0
Balance		8.25	6.75	-4.95	-4.5	-5.55		
C.O	4.13		-2.48	3.38				
Balance		1.36	1.12	-1.12	-1.01	-1.25		
C.O	0.68		0.56	0.56				
Balance		0.31	0.25	-0.18	-0.17	-0.21		
C.O	0.16		-0.09	0.13				
Balance		0.05	0.04	-0.04	-0.04	-0.05		
C.O	0.03							
Final moments	**5**	**9.97**	**-9.97**	**12.78**	**-5.72**	**-7.06**	**0**	**0**

Step: 4 – BMD



8. Analyze the frame shown in figure by moment distribution method and draw BMD and SFD



Step: 1 - Fixed end moments

$$M_{FAB} = -\frac{20x5^2}{12} = -41.67 \text{ KNM}$$

$M_{FBA} = +41.67$ KNM

$M_{FBD} = M_{FDB} = 0$

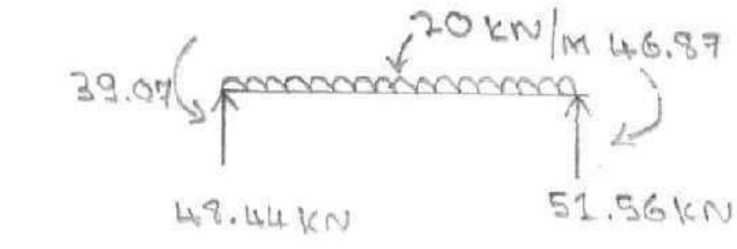
$M_{FBC} = -50 \times 1 = -50$ KNM

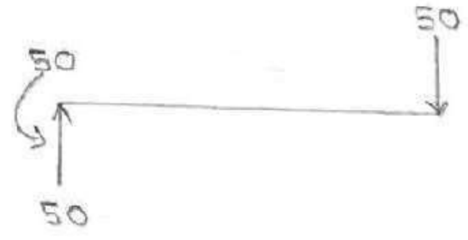
Step:2 - Distribution factor

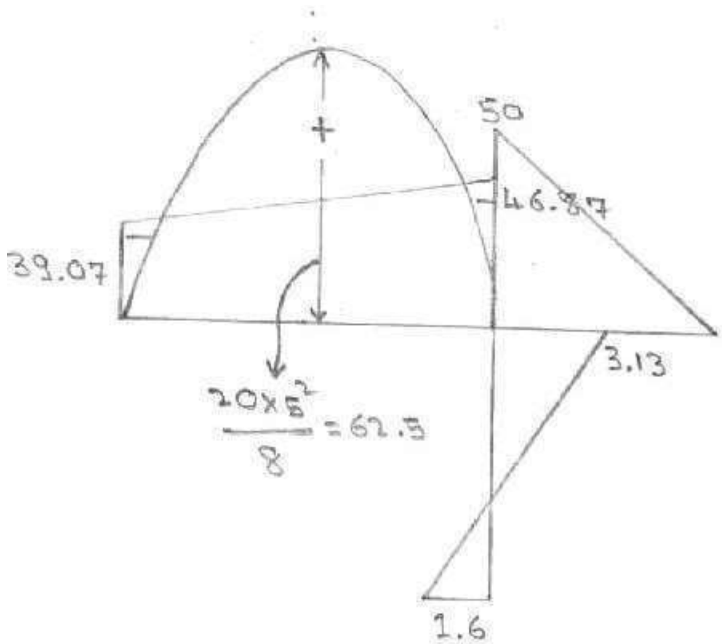
Jt.	Member	K	ΣK	$DF = \frac{K}{\sum K}$
B	BA	2I/5 = 0.4I		0.62
	BC	0	0.65I	0
	BD	I/4 = 0.25 I		0.38

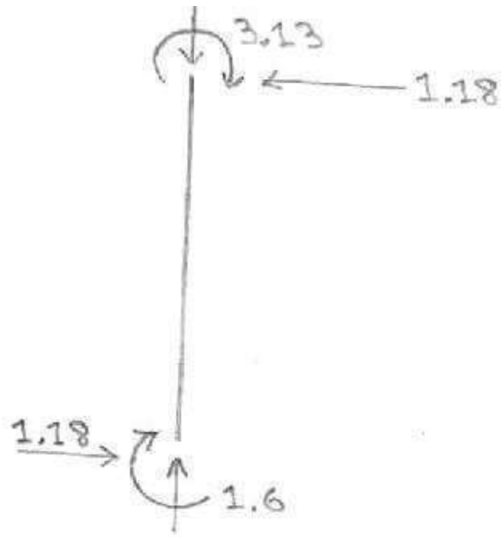
Step: 3 - Moment Distribution

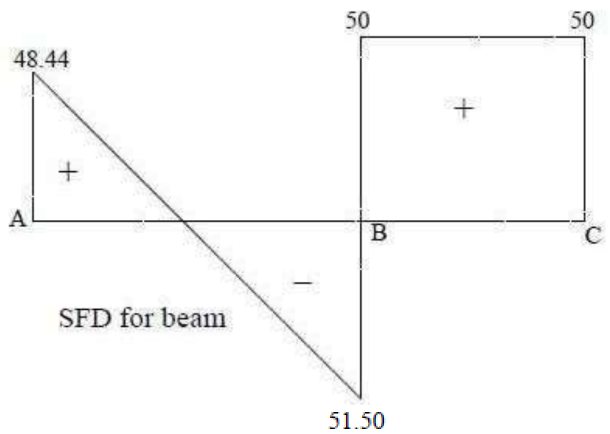
Jt	A	B			D
Member	AB	BA	BC	BD	DB
D.F	01	0.62	0	0.38	0
FEM	-41.67	41.67	-50	0	0
Balance		5.2	0	3.13	
C.O	2.6				1.6
Final moments	**-39.07**	**46.87**	**-50**	**3.13**	**1.6**

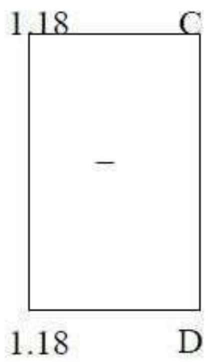








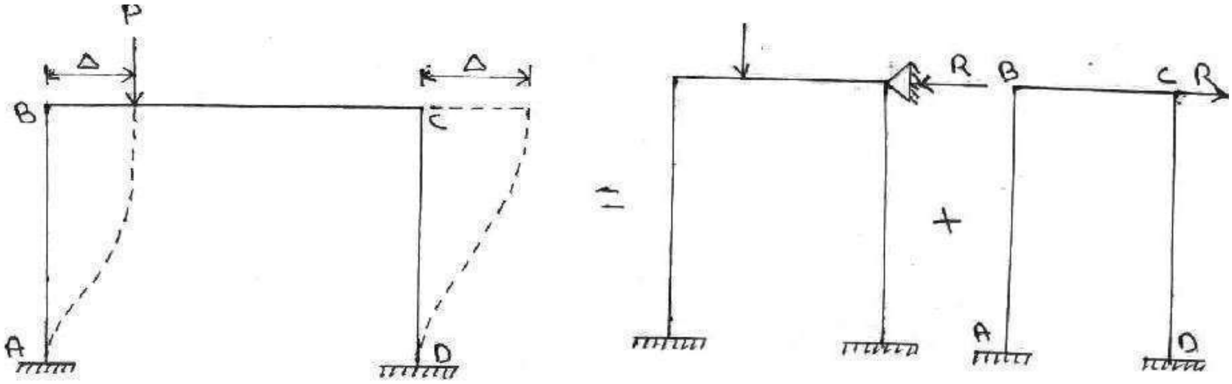




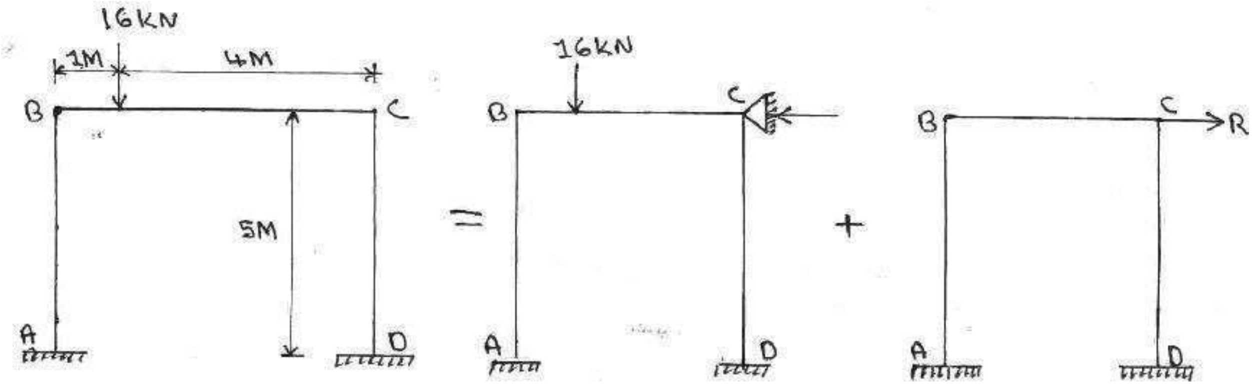
Transvers shear force diagram for

Moment distribution method for frames with side sway:

- Frames that are non symmetrical with reference to material property or geometry (different lengths and I values of column) or support condition or subjected to nonsymmetrical loading have a tendency to side sway.



9. Analyze the frame shown in figure by moment distribution method. Assume EI is constant.



Non Sway Analysis:

- First consider the frame held from side sway as shown in figure.

Step: 1 - Fixed end moments

$$M_{FAB} = M_{FBA} = M_{FCD} = M_{FDC} = 0$$

$$M_{FBC} = -\frac{16\times1\times4^2}{5^2} = -10.24 \text{ kNm}$$

$$M_{FCB} = \frac{16\times1^2\times4}{5^2} = 2.56 \text{ kNm}$$

Step:2 - Distribution factor

Jt.	Member	Relative stiffness K	ΣK	DF = $\frac{K}{\sum K}$
B	BA BC	I/5 = 0.2 I I/5 = 0.2 I	0.4 I	0.5 0.5
C	CB CD	I/5 = 0.2I I/5 = 0.2I	0.4I	0.5 0.5

Step: 3 - Moment Distribution

Joint	A	B		C		D
Member	AB	BA	BC	CB	CD	DC
D.F	0	0.5	0.5	0.5	0.5	0
FEM	0		-10.24	2.56	0	0
Balance		5.12	5.12	1.28	-1.28	
CO	2.56		-0.64	2.56		-0.64
Balance		0.32	0.32	-0.08	-0.08	
CO	0.16		-0.64	0.16		-0.64
Balance		0.32	0.32	-0.08	-0.08	
C.O	0.16		-0.04	0.16		-0.04
Balance		0.02	0.02	-0.08	-0.08	
C.O	0.01					-0.04
Final moments	**2.89**	**5.78**	**-5.78**	**2.72**	**-2.72**	**-1.36**

FBD of columns:

By seeing of the FBD of columns

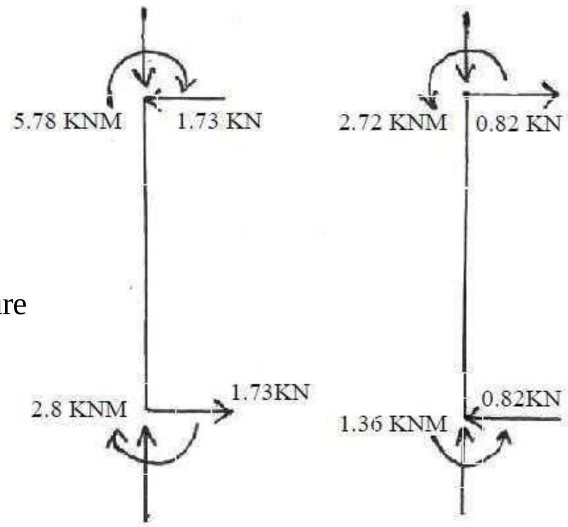
R = 1.73 – 0.82

(Using $\Sigma F_x = 0$ for entire frame) = 0.91 KN (⟵)

Now apply

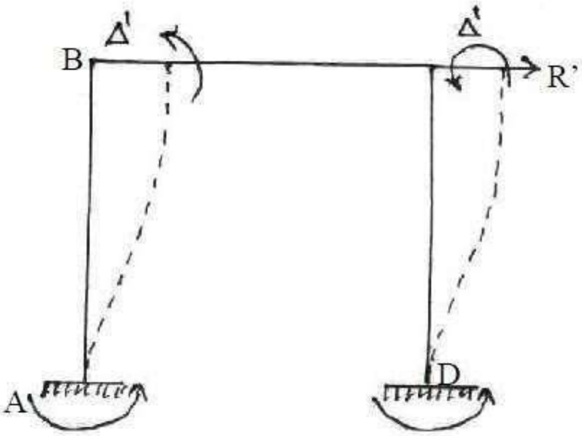
R = 0.91 kN acting opposite as shown in figure

for the sway analysis.



ii) Sway analysis:

- For this we will assume a force R′ is applied at C causing the frame to deflect Δ" as shown in figure



Since both ends are fixed, columns are of same length & I and assuming joints B & C are temporarily restrained from rotating and resulting fixed end moment are

$$M'_{BA} = M'_{AB} = M'_{CD} = M'_{DC} = \frac{6EI\Delta'}{l^2}$$

Assume $M^1_{BA} = -100$ kNm

$\therefore\ M'_{AB} = M'_{CD} = M'_{DC} = -100$ kNm

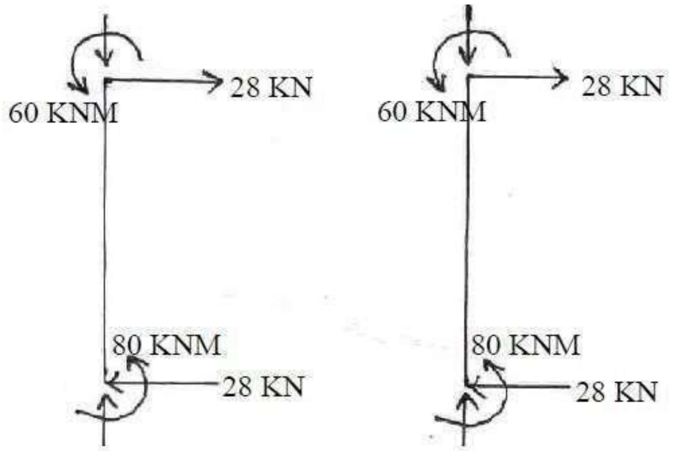
Moment distribution table for sway analysis:

Joint	A	B		C		D
Member	AB	BA	BC	CB	CD	DC
D.F	01	0.5	0.5	0.5	0.5	0
FEM	−100	−100	0	0	−100	−100
Balance		50	50	50	50	
CO	25		25	25		25
Balance		−12.5	−12.5	−12.5	−12.5	
CO	-6.25		−6.25	-6.25		−6.25
Balance		3.125	3.125	3.125	3.125	
C.O	1.56		1.56	1.56		1.56
Balance		−0.78	−0.78	−0.78	−0.78	
C.O	-0.39		−0.39	−0.39		0.39
Balance		0.195	0.195	0.195	0.195	
C.O	0.1					0.1
Final moments	**- 80**	**- 60**	**60**	**60**	**– 60**	**– 80**

FBD of columns:

Using $\sum F_x = 0$ for the entire frame

$R' = 28 + 28 = 56$ KN $(\rightarrow)$



- Hence R'' = 56KN creates the sway moments shown in above moment distribution table. Corresponding moments caused by R = 0.91KN can be determined by proportion.
- Thus final moments are calculated by adding non sway moments and sway moments calculated for R = 0.91KN, as shown below

$$M_{AB} = 2.89 + \frac{0.91}{56}(-80) = 1.59 \text{ kNm}$$

$$M_{BA} = 5.78 + \frac{0.91}{56}(-60) = 4.81 \text{ kNm}$$

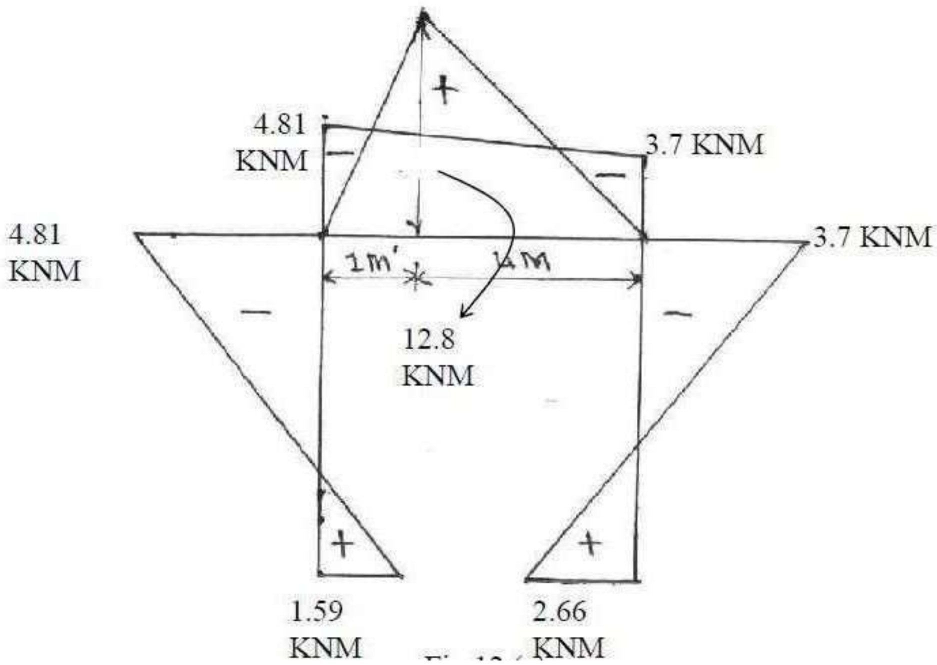
$$M_{BC} = -5.78 + \frac{0.91}{56}(60) = -4.81 \text{ kNm}$$

$$M_{CB} = 2.72 + \frac{0.91}{56}(60) = 3.7 \text{ kNm}$$

$$M_{CD} = -2.72 + \frac{0.91}{56}(-60) = -3.7 \text{ kNm}$$

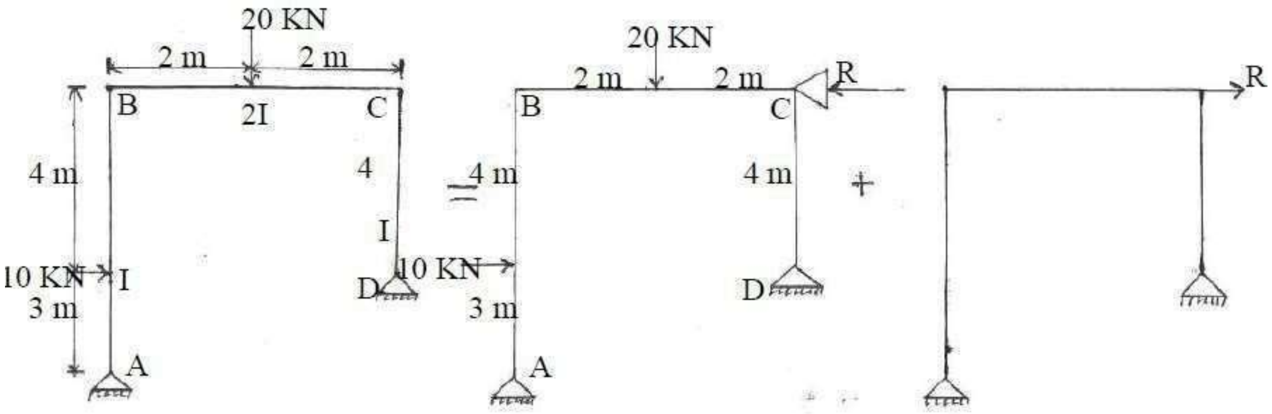
$$M_{DC} = -1.36 + \frac{0.91}{56}(-80) = -2.66 \text{ kNm}$$

BMD:



Moment distribution method for frames with side sway:

1. Analysis the rigid frame shown in figure by moment distribution method and draw BMD



i) Non Sway Analysis:

First consider the frame held from side sway as shown in figure 2

FEM calculation:

$$M_{FAB} = -\frac{10\times3\times4^2}{7^2} = -9.8\,KNM$$

$$M_{FBA} = +\frac{10\times3^2\times4}{7^2} = 7.3\,KNM$$

$$M_{FBC} = -\frac{20\times4}{8} = -10\,KNM$$

$$M_{FCB} = +10KNM$$

$$M_{FCD} = M_{FDC} = 0$$

Distribution Factor:

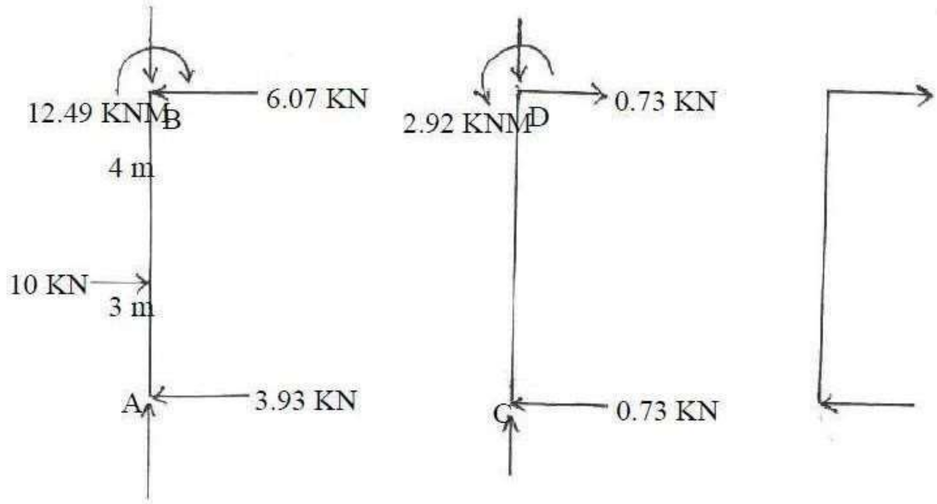
Joint	Member	Relative stiffness k	Σk	DF = $\frac{K}{\sum K}$
B	BA BC	$\frac{3}{4}\times\frac{I}{7} = 0.11I$ 2I/4 = 0.5I	0.61 I	0.18 0.82
C	CB CD	2I/4 = 0.5I $\frac{3}{4}\times\frac{I}{4} = 0.19\,I$	0.69 I	0.72 0.28

Moment distribution for non sway analysis:

Joint	A	B		C		D
Member	AB	BA	BC	CB	CD	DC
D.F	1	0.18	0.82	0.72	0.28	1
FEM	-9.8	7.3	-10	10	0	0
Release jt. 'D'	+9.8					
CO		4.9				
Initial moments	0	12.2	-10	10	0	0
Balance		-0.4	-1.8	-7.2	-2.8	
CO			-3.6	-0.9		
Balance		0.65	2.95	0.65	0.25	
C.O			0.33	1.48		
Balance		-0.06	-0.27	-1.07	-0.41	
C.O			-0.54	-0.14		
Balance		0.1	0.44	0.1	0.04	
Final moments	**0**	**12.49**	**-12.49**	**2.92**	**-2.92**	**0**

FBD of columns:

- FBD of columns AB & CD for non-sway analysis is shown in figure

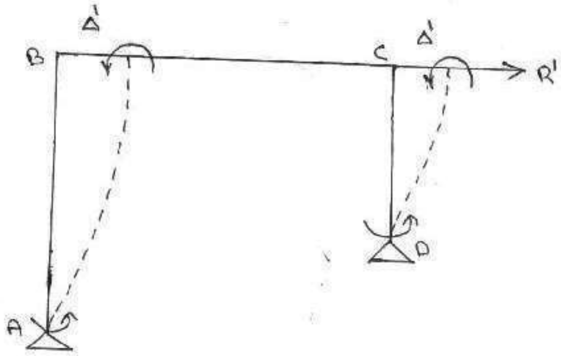


Applying $\Sigma F_x = 0$ for frame as a
Whole, R = 10 – 3.93 – 0.73
= 5.34 kN (←)

- Now apply R = 5.34KN acting opposite as shown in figure for sway analysis

ii) Sway analysis:

- For this we will assume a force R′ is applied at C causing the frame to deflect Δ′ as shown in figure.



Since ends A & D are hinged and columns AB & CD are of different lengths

$M'_{BA} = -3EI\Delta'/L^2_1$

$M'_{CD} = -3EI\Delta'/L^2_2$

$$\therefore \frac{M^1_{BA}}{M^1_{CD}} = \frac{3EI\ \Delta'/l_1^2}{3EI\ \Delta^1/l_2^2} = \frac{l_2^2}{l_1^2} = \frac{4^2}{7^2} = \frac{16}{49}$$

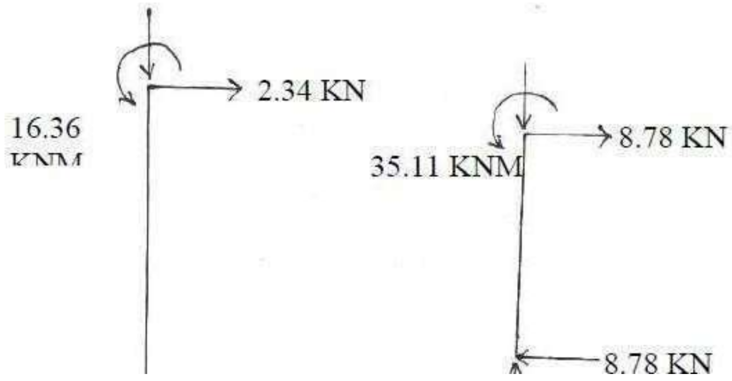
Assume $M^1_{BA} = -16\text{kNm}\ \&\ M'_{AB} = 0$

$\&\ M^1_{CD} = -49\text{kNm}\ \&\ M'_{DC} = 0$

Moment distribution table for sway analysis:

Joint	A	B		C		D
Member	AB	BA	BC	CB	CD	DC
D.F	1	0.18	0.82	0.72	0.28	1
FEM	0	-16	0	0	-49	0
Balance		2.88	13.12	35.28	13.72	
CO			17.64	6.56		
Balance		-3.18	-14.46	-4.72	-1.84	
CO			-2.36	-7.23		
Balance		0.42	1.94	5.21	2.02	
C.O			2.61	0.97		
Balance		-0.47	-2.14	-0.7	-0.27	
C.O			0.35	-1.07		
Balance		0.06	0.29	0.77	0.3	
C.O			0.39	0.15		
Balance		-0.07	-0.32	-0.11	-0.04	
Final moments	**0**	**-16.36**	**16.36**	**35.11**	**-35.11**	**0**

FBD of columns AB & CD for sway analysis moments is shown in fig.



Using $\Sigma f_x = 0$ for the entire frame
$R' = 11.12$ kN ($\rightarrow$)

Hence $R' = 11.12$ kN creates the sway moments shown in the above moment distribution table. Corresponding moments caused by $R = 5.34$ kN can be determined by proportion.

Thus final moments are calculated by adding non-sway moments and sway moments determined for R = 5.34 KN as shown below.

$M_{AB} = 0$

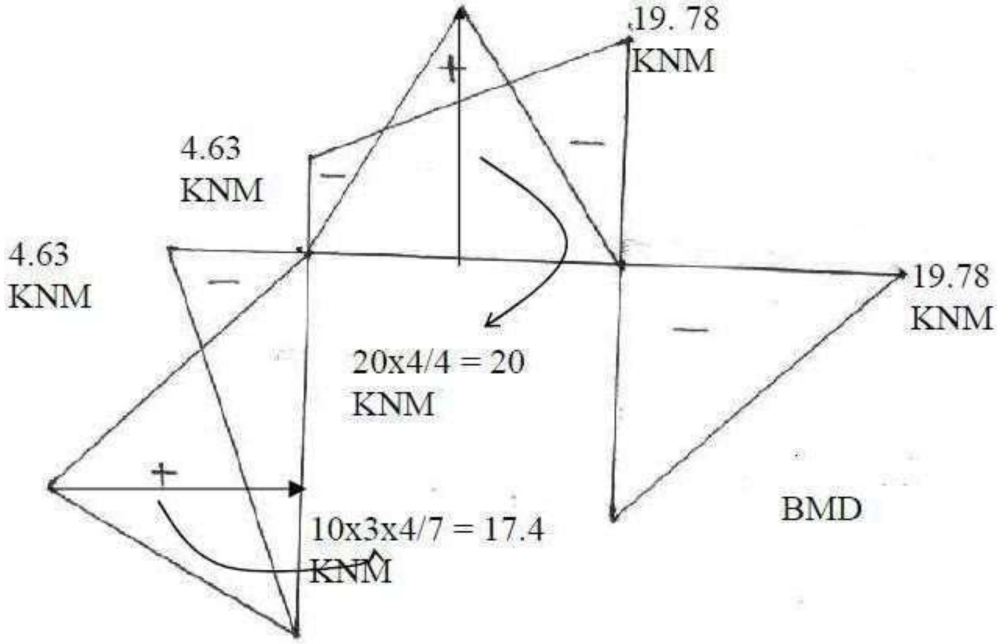
$M_{BA} = 12.49 + \frac{5.34}{11.12}(-16.36) = 4.63$ kNm

$M_{BC} = -12.49 + \frac{5.34}{11.12}(16.36) = -4.63$ KNM

$M_{CB} = 2.92 + \frac{5.34}{11.12}(35.11) = 19.78$ kNm

$M_{CD} = -2.92 + \frac{5.34}{11.12}(-35.11) = -19.78$ KNM

$M_{DC} = 0$



UNIT-III

ARCHES

Arches as structural forms – Examples of arch structures – Types of arches – Analysis of three hinged, two hinged and fixed arches, parabolic and circular arches – Settlement and temperature effects.

Introduction:

- Mainly three types of arches are used in practice: three-hinged, two-hinged and hingeless arches.
- In 19th century, three-hinged arches were commonly used for the long span structures.
- Then development in structural analysis, for long span structures starting from late nineteenth century engineers adopted two-hinged and hingeless arches.
- Two-hinged arch is the statically indeterminate structure to degree one.
- Usually, the horizontal reaction is treated as the redundant and is evaluated by the method of least work.

Arch:

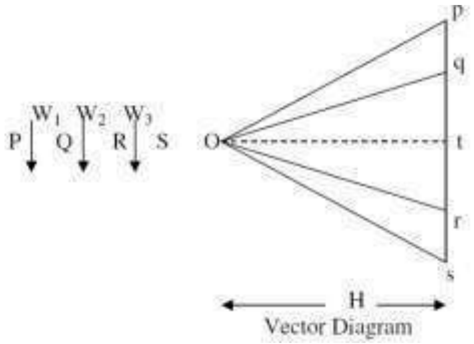
- An arch is a curved beam or structure in vertical plane and subjected to transverse loads which act on the convex side of the curve and re-sights the external loads by virtue of thrust.
- It is subjected to three restraining forces i.e.,
    - Thrust
    - Shear force
    - Bending Moment

What is an arch? Explain.

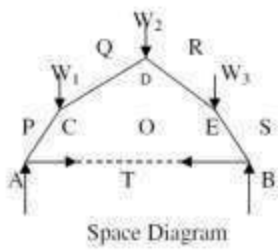
- An arch is defined as a curved girder, having convexity upwards and supported at its ends.
- The supports must effectively arrest displacements in the vertical and horizontal directions.
- Only then there will be arch action.

What is a linear arch?

- If an arch is to take loads, say W_1, W_2, and W_3 (fig) and a Vector diagram and funicular polygon are plotted as shown, the funicular polygon is known as the linear arch or theoretical arch.



Vector Diagram

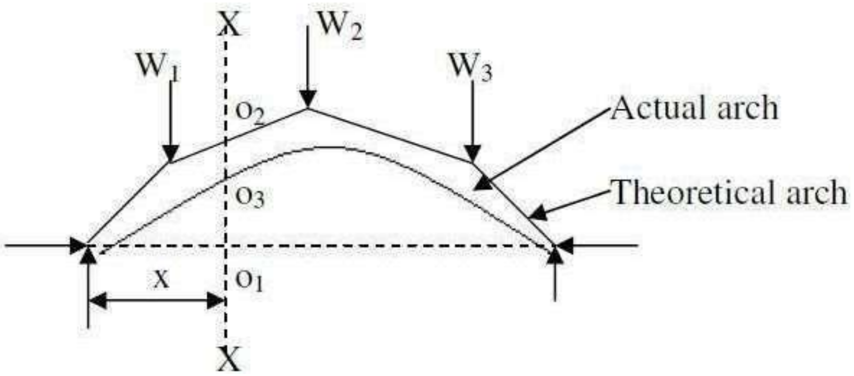


Space Diagram

- The polar distance „ot‟ represents the horizontal thrust.
- The links AC, CD, DE, and EB will be under compression and there will be no bending moment.
- If an arch of this shape ACDEB is provided, there will be no bending moment.
- For a given set of vertical loads W1, W2…..etc., we can have any number of linear arches depending on where we choose „O‟ or how much horizontal thrust (or) we choose to introduce.

State Eddy's theorem.

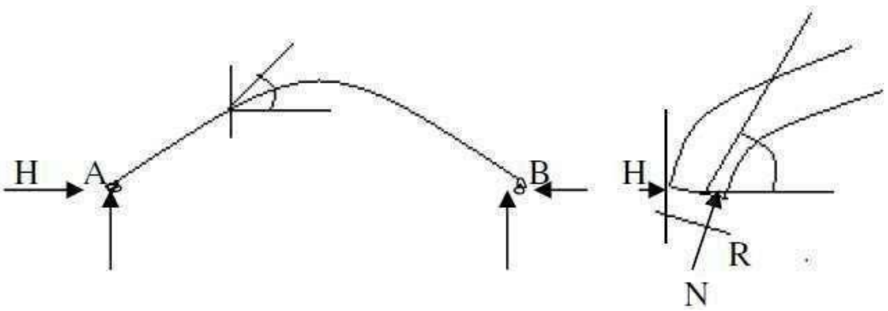
- Eddy‟s theorem states that "The bending moment at any section of an arch is proportional to the vertical intercept between the linear arch (or theoretical arch) and the centre line of the actual arch."
- BM_x = Ordinate O_2O_3 x scale factor



What is the degree of static indeterminacy of a three hinged parabolic arch?

- For a three hinged parabolic arch, the degree of static indeterminancy is zero.
- It is statically determinate.

Explain with the aid of a sketch, the normal thrust and radial shear in an arch rib.



- Let us take a section X of an arch. (fig (a)).
- Let q be the inclination of the tangent at X.
- If H is the horizontal thrust and V the vertical shear at X, from the free body of the RHS of the arch, it is clear that V and H will have normal and radial components given by,

$$N = H\cos\Theta + V\sin\Theta$$

$$R = V\cos\Theta - H\sin\Theta$$

Difference between the basic action of an arch and a suspension cable

- An arch is essentially a compression member which can also take bending moments and shears.
- Bending moments and shears will be absent if the arch is parabolic and the loading uniformly distributed.
- A cable can take only tension. A suspension bridge will therefore have a cable and a stiffening girder.
- The girder will take the bending moment and shears in the bridge and the cable, only tension.
- Because of the thrusts in the cables and arches, the bending moments are considerably reduced.
- If the load on the girder is uniform, the bridge will have only cable tension and no bending moment on the girder.

Distinguish between two hinged and three hinged arches

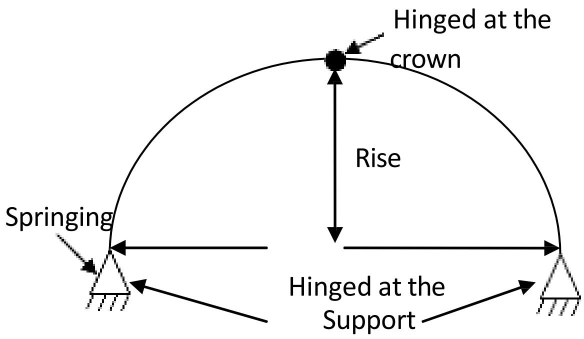
Sl. No	*Two hinged arches*	*Three hinged arches*
1	Statically indeterminate to first degree	Statically determinate
2	Might develop temperature stresses	Increase in temperature causes increase in Central rise. No stresses.
3	Structurally more efficient	Easy to analyse. But in costruction, the central hinge may involve additional expenditure.
4	Will develop stresses due to sinking of supports	Since this is determinate, no stresses due to support sinking.

Types of Arches

a). According to the support conditions (structural behaviour arches) or hinges

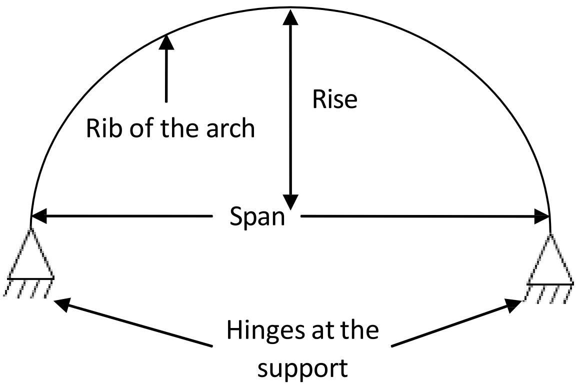
i. *Three hinged arch*

- Hinged at the supports and the crown
- A 3-hinged arch is a statically determinate structure.



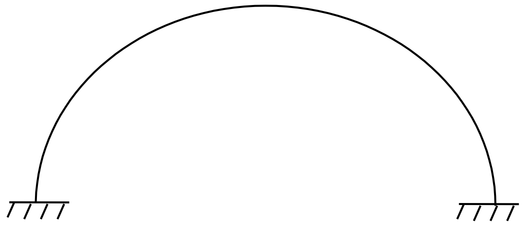
ii. *Two hinged arch*

- Hinged only at the support
- It is an indeterminate structure of degree of indeterminacy equal to 1



iii. ***Single hinged arch***

iv. ***Fixed arch (or) hingeless arch***



- The supports are fixed
- It is a statically indeterminate structure.
- The degree of indeterminancy is 3

b). According to their shapes

v. Circular or curved or segmental arch

vi. Parabolic arch

vii. Elliptical arch

viii. Polygonal arch

c). According to their basis of materials

i. Steel arches,

ii. Reinforced concrete arches,

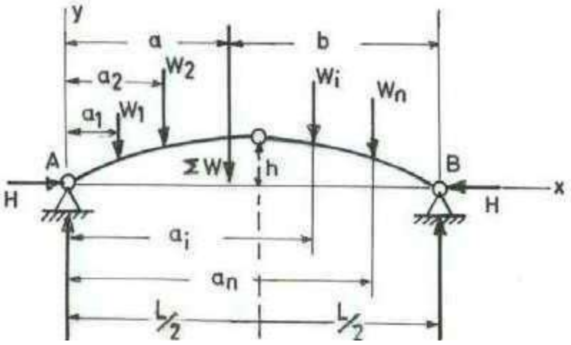
iii. Masonry arches (Brick or Stone) etc.,

d). According to their space between the loaded area and the rib arches

i. Open arch

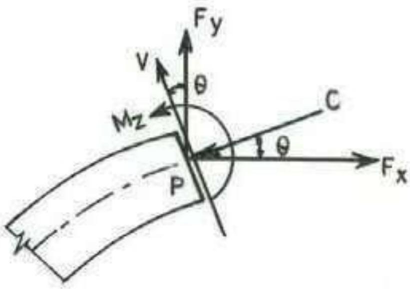
ii. Closed arch (solid arch).

Three Hinged Arch



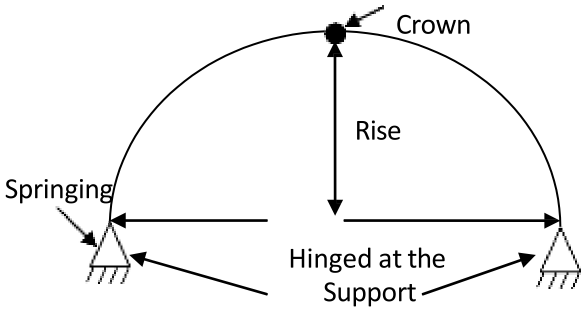
- Three hinged arch is statically determinate.
- Third hinge at crown and the other two hinges at each abutments
- Mostly used for long span bridges

Analysis of three Hinged Parabolic Arch



- Bending moment at the crown hinge is zero
- Arch have two reaction at support (One horizontal & one vertical)
- Need for four equation to solve and find the unknown reaction.

✦ We can use three static equilibrium conditions and in addition to that the B.M. at the crown hinge is equal to zero.



For Symmetric Parabolic Arch:

1. Rise:

$$y = \frac{4h}{L^2} x (L - x)$$

Where,

y_c = r = Radius (or) Rise of arch

L = Length of Arch or Span

2. Internal forces (Fx, Fy & Mz)

a. Normal Thrust (N_x)

N_x = $V_x \sin\Theta + H \cos\Theta$

b. Radial Shear (R_x)

R_x = $V_x \sin\Theta - H \cos\Theta$

c. Slope of arch (Θ)

θ = $\tan^{-1} [(4h/L^2) (L - 2x)]$

d. Resultant (R)

R_A = $\sqrt{(V_A^2 + H_A^2)}$

Where,

F_x or R_x = shear force in the arch

F_y or N_x = thrust in the arch

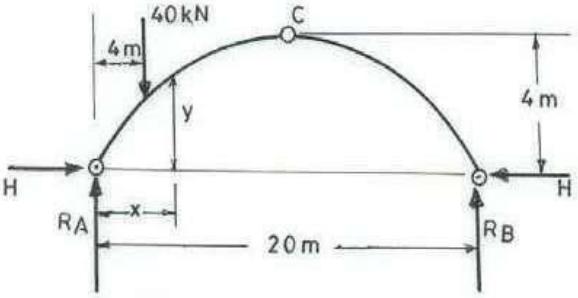
θ = Slope of arch axis at P.

V = Shear at P

C = Thrust at P

M = Bending moment at P

1. ***A three hinged parabolic arch of 20 m span and 4 m central rise as shown in figure carries a point load of 40 kN at 4 m horizontally from left support. Compute BM, SF and AF at load point. Also determine maximum positive and negative bending moments in the arch and plot the bending moment diagram.***



$$y = \frac{4h}{L^2} x (L - x) = \frac{4 \times 4}{400} x (20 - x)$$

$$y = \frac{x}{25} (20 - x)$$

$$R_B = \frac{4}{20} \times 40 = 8 \text{ kN}, R_A = \frac{16}{20} \times 40 = 32 \text{ kN}$$

$M_C = 0$, $4H = 32 \times 10 - 40 \times 6 = 80$, $H = 20$ kN

$0 \leq x \leq 4$ m

$$M_x = 32x - 20 \frac{x}{25} (20 - x) = 16x + \frac{4}{5} x^2$$

BENDING MOMENT DIAGRAM

$$x = 4, M_x = 16 \times 4 + \frac{4 \times 16}{5} = 76.8 \text{ kNm}$$

$4 \text{ m} \leq x \leq 20$ m

$$M_x = 32x - 20 \frac{x}{25} (20 - x) - 40 (x - 4) = 160 - 24x + \frac{4}{5} x^2$$

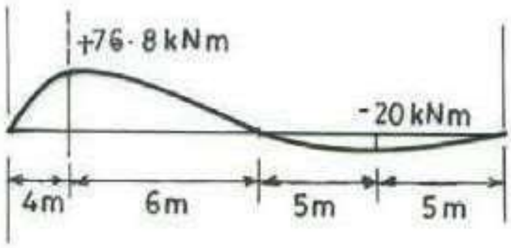
$x = 4$, $M_x = 76.8$ kNm (check)

$x = 10$, $M_x = 160 - 240 + 80 = 0$

$$x = 15, M_x = 160 - 24 \times 15 + \frac{4}{5} \times 225 = -20 \text{ kNm}$$

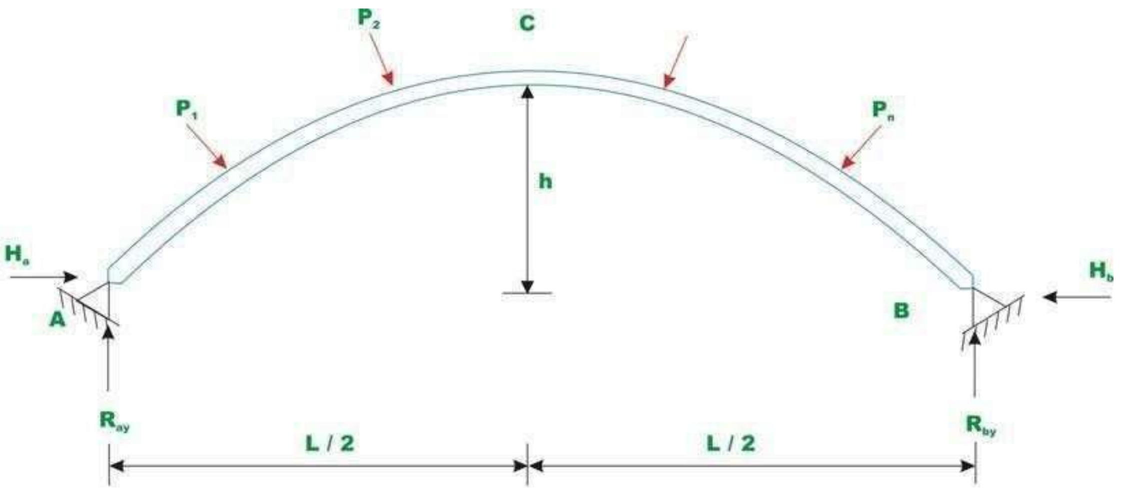
$x = 20$, $M_x = 0$ (ok)

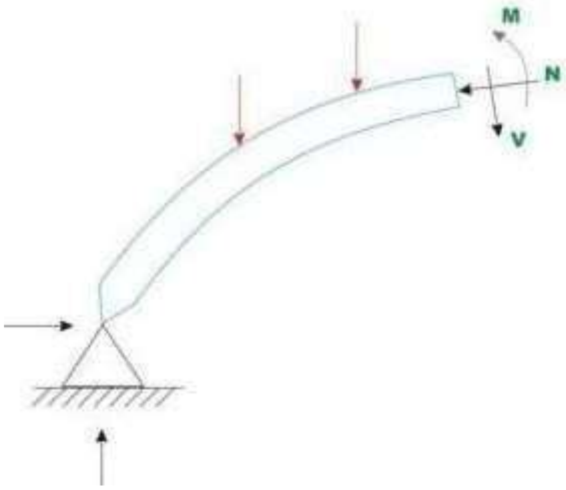
B.M.D



Analysis of two-hinged arch

- A typical two-hinged arch is having four unknown reactions, but there are only three equations of equilibrium available.
- Hence, the degree of statically indeterminacy is one for two-hinged arch.





Rib-shortening in the case of arches.

- In a two hinged arch, the normal thrust which is a compressive force along the axis of the arch will shorten the rib of the arch.
- This in turn will release part of the horizontal thrust.
- Normally, this effect is not considered in the analysis (in the case of two hinged arches).
- Depending upon the importance of the work we can either take into account or omit the effect of rib shortening.
- This will be done by considering (or omitting) strain energy due to axial compression along with the strain energy due to bending in evaluating H.

Strain energy due to bending (U_b)

$$U_b = \int_0^s \frac{M^2}{2EI} ds$$

Where,

M	=	*Bending moment*
E	=	*Young's modulus of the arch material*
I	=	*Moment of inertia of the arch cross section*
s	=	*Length of the centreline of the arch*

Strain energy due to axial compression (U_a)

$$U_a = \int_0^s \frac{N^2}{2AE} ds$$

Where,

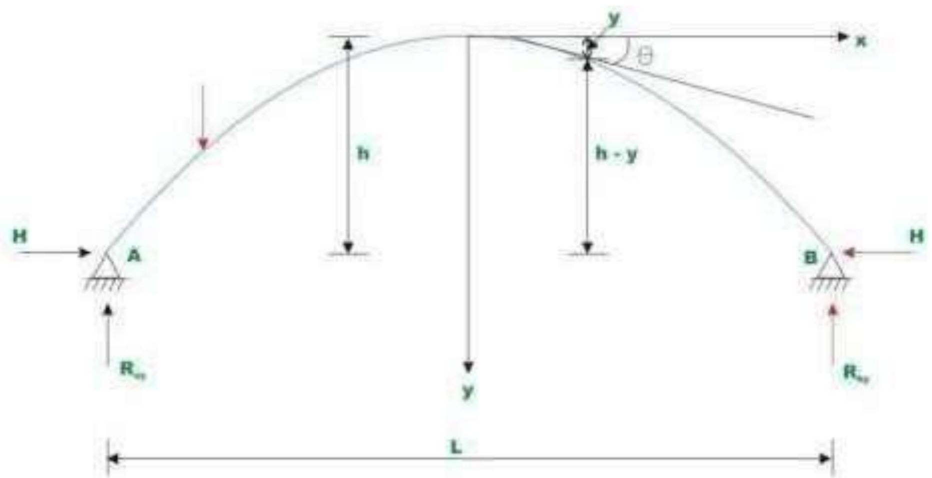
M	=	Bending moment
N	=	Axial compression.
A	=	*Cross sectional area of the arch*
E	=	*Young's modulus of the arch material*
s	=	*Length of the centreline of the arch*

Total strain energy of the arch

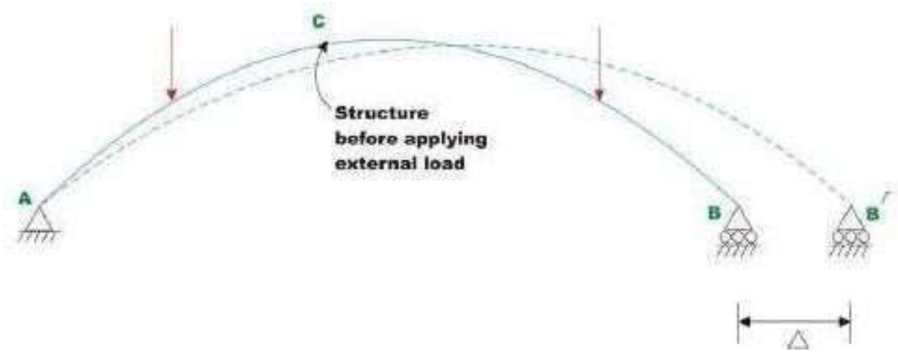
$$U = \int_0^s \frac{M^2}{2EI} ds + \int_0^s \frac{N^2}{2AE} ds$$

Symmetrical two hinged arch

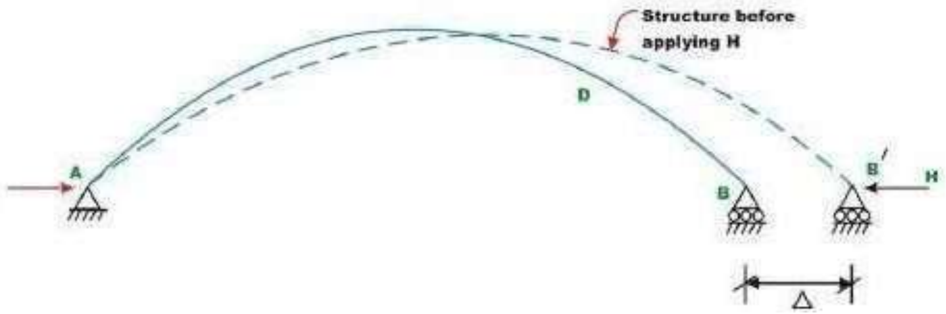
- Consider a symmetrical two-hinged arch as shown in figure.



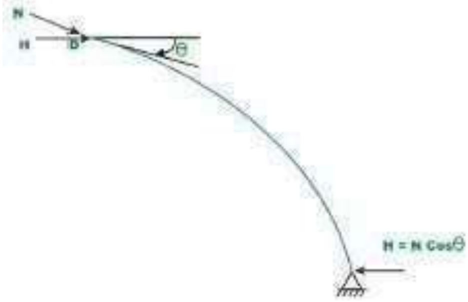
- Let „C" at crown be the origin of co-ordinate axes.
- Now, replace hinge at 'B' with a roller support.
- Then we get a simply supported curved beam figure as shown in below.



- Since the curved beam is free to move horizontally, it will do so as shown by dotted lines.
- Let M_o and N_o be the bending moment and axial force at any cross section of the simply supported curved beam.
- Since, in the original arch structure, there is no horizontal displacement, now apply a horizontal force „*H'* as shown in figure.



- The horizontal force „*H'* should be of such magnitude, that the displacement at „*B'* must vanish.



Bending moment at any cross section of the arch **M = M_o – H (h-y)**

The axial compressive force at any cross section **N = N_o + H cos*θ***

Where,

θ ⟶ the angle made by the tangent at D with horizontal

Substituting the value of *M* and N in the equation

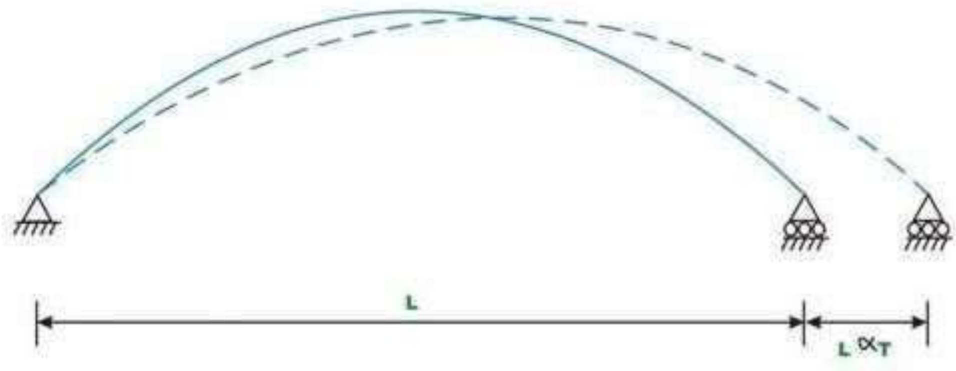
$$\frac{\partial U}{\partial H} = 0 = -\int_0^s \frac{M_0 - H(h-y)}{EI}(h-y)ds + \int_0^s \frac{N_0 + H\cos\theta}{EA}\cos\theta\, ds$$

$$H = \frac{\int_0^s M_0 \tilde{y}\, ds}{\int_0^s \tilde{y}^2\, ds}$$

Temperature effect

- Consider an unloaded two-hinged arch of span *L*.
- When the arch undergoes a uniform temperature change of *T* °*C*, then its span would increase by *α L T* if it were allowed to expand freely.
- α is the co-efficient of thermal expansion of the arch material.

- Since the arch is restrained from the horizontal movement, a horizontal force is induced at the support as the temperature is increased.



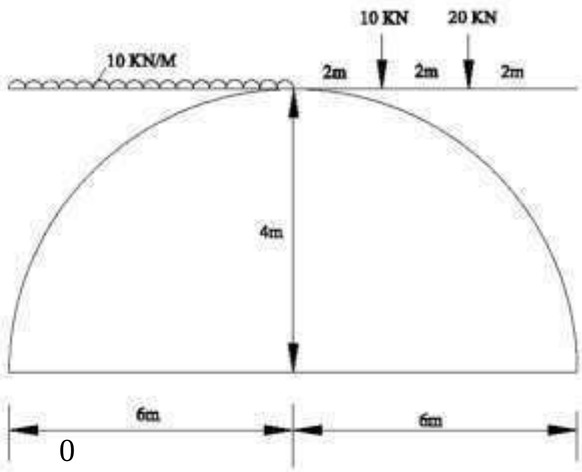
Analysis of 3-hinged arches

- It is the process of determining external reactions at the support and internal quantities such as normal thrust, shear and bending moment at any section in the arch.

Procedure to find reactions at the supports

- *Sketch the arch with the loads and* reactions at the support.
- Apply equilibrium conditions namely $\sum F_x = 0$, $\sum F_y = 0$ and $\sum M = 0$
- Apply the condition that BM about the hinge at the crown is zero (Moment of all the forces either to the left or to the right of the crown).
- Solve for unknown quantities.

1. ***Find the BM,RS,NT catch 4m from left hand side,2m from right hand side of the three hinged parabolic arch shown in fig.***



Step1: Find V_A, V_B, H_A, H_B

Take M @ B = 0

$V_A \times 12 + H_A \times 0 - 10 \times 16(6/2+6) - 20 \times 4 + 20 \times 2 = 0$

V_A = 55 KN

Take M @ A = 0

$-V_B \times 12 + H_B \times 0 + 20 \times 0 + 20 \times 8 + (10 \times 6^2/2) = 0$

V_B = 45 KN

H_A (LHS)

$V_A \times 6 - (H_A \times 4) - (10 \times 6^2/2) = 0$

H_A = 37.5 KN

H_B (RHS)

$-V_B\times6+H_B\times4-20\times4+20\times2 = 0$

H_B = 37.5 KN

(i) BM, RS, NT (4m from LHS)

$Y_D = 4r/L^2(Lx-x^2)$

$= 4\times4/12^2(12\times4-4^2)$

Y_D = 3.56 m

Bending moment

$M@D = (V_A\times4)-(H_A\times3.56)-(10\times4^2/2)$

M_D = 6.5 KNm

Normal thrust

$NT = V_x Sin\theta + HCos\theta$

$V_x = V_A-(10\times4)$

$= 55-40$

$= 15$ KN

$\theta = 4r/L^2 (L-2x)$

$= 25.46$

$NT = 15\times\cos 25.46-37.5\times\sin 25.46$

$= 40.3$ KN

Radial shear

$RS = V_x\cos\theta - H\sin\theta$

$= 15\times\cos 25.46 - 37.5 \times \sin 25.46$

$= 2.58$ KN

(ii) BM, RS, NT (2m from RHS)

$Y_E = 2.22$ m

$V_X = 25$ KN

Bending moment

$M@E = (-V_B\times2) + H_B\times2.22$

$M@E = -6.75$KNm

Normal thrust

$NT = V_x Sin\theta + HCos\theta$

$V_x = V_B-20$

$= 45-20$

$= 25$ KN

$\theta = 4r/L^2 (L-2x) = 50.92$

NT = 25×cos 55.92-37.5×sin 55.92

= 43KN

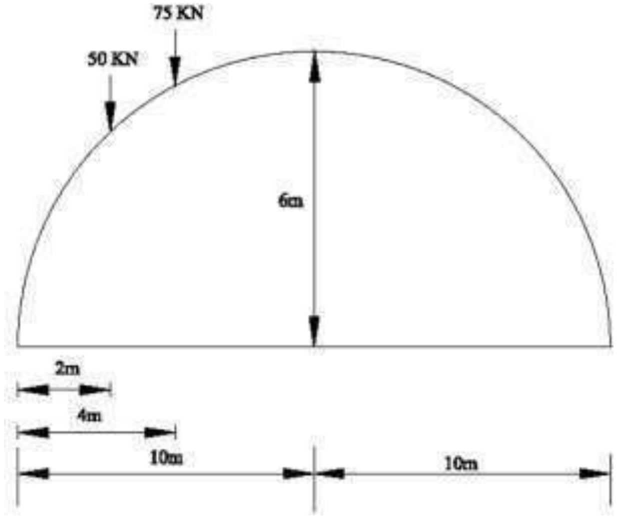
Radial shear

RS = $V_x \cos\theta$ - $H\sin\theta$

= 25×cos 55.92 – 37.5 × sin 55.92

= -13.33 KN

2. *Three hinged circular arch, a find support reaction, BM, RS, NT at 4m from L.H.S and 5m from R.H.S.*



Solution: Find V_A, V_B, H_A, H_B

Take M @ B = 0

V_A×20-50×18-75×16 = 0

V_A = 105 KN

Take M @ A=0

V_B = 20 KN

H_A(LHS)

V_A×10-(H_A×6)- (50×8) - (75×6) = 0

H_A = 33.33 KN

H_B (RHS)

H_B = 33.33 KN

Find Y_D

r (2R-r) = $L^2/4$

L = 20m, r = 6m

6(2R-r) = $20^2/4$

R = 11.33 m

R^2 = $x'^2+(R-r+Y_D)^2$ x′=6m

11.33^2 = $6^2+(11-6+Y_D)^2$

9.61 = 5.33+ Y_D

Y_D=4.28 m

θ = $\sin^{-1}(x'/R)$

= $\sin^{-1}(6/11.33)$

= 31.98°

(i) BM, RS, NT (4m from LHS)

Bending moment

$M@D = (V_A \times 4)-(50\times 2)-(H_A \times Y_D)$

$= 117.35$ KNm

Normal thrust

$NT = V_x Sin\theta + HCos\theta$

$V_x = V_A-(75+50)$

$= 20$ KN

$NT = 20Sin31.98 + 33.33Cos31.98$

$= 17.98$ KN

Radial shear

$RS = V_x \cos\theta - H \sin\theta$

$= 20\times\cos 31.98 - 33.33\times \sin31.98$

$= - 34.61$ KN

(ii) BM,RS,NT(6m from RHS)

$x' = 5$ m

$\theta = 26.18$

Bending moment

$Y_E = 4.83$

$M@E = (VB\times 5)+(H_B\times Y_E)$

$= (20\times 5) + (33.33\times 4.83)$

$= 60.98$ KNm

Normal thrust

$NT = V_x Sin\theta + H Cos\theta$

$V_x = 20$ KN

$NT = 20 Sin26.18 + 33.33 Cos26.18$

$= 38.73$ KN

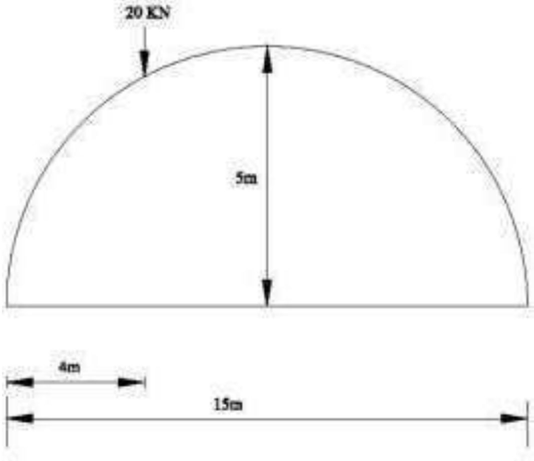
Radial shear

$RS = V_x \cos\theta - H \sin\theta$

$= 20\times\cos 26.18 - 33.33\times \sin26.18$

$= - 3$ KN

3. ***A two hinged parabolic arch of span 15m and a point load of 20 KN at a distance of 4m from L.H.S. Find the BM, RS, NT 4m from L.H.S and 3m from R.H.S. since r = 5m.***



Solution: Find V_A,V_B

Take M@B = 0

$V_A \times 15 - 20 \times 11 = 0$

$V_A = 14.67$ KN

Take M@A = 0

$-V_B \times 15 + 20 \times 4 = 0$

$V_B = 5.3$ KN

$H = {}_0\int^l (\mu\, y\, dx)/(18r^2L/15)$

$\mu_1 = 14.67\, x_1$

$\mu_2 = -5.33x_2+80$

Substitute the μ_2,μ_1 values to above equation and we get

$H = 11.8$ KN

(i) BM,RS,NT(4m from LHS)

Bending moment

$Y_D = 3.9$ m

$M@D = (V_A \times 4)-(20 \times 0)-(H_A \times Y_D)$

$= 12.62$ KNm

Normal thrust

$NT = V_x \sin\theta + H\cos\theta$

$V_x = V_A - 20$

$= -5.33$ KN

$\theta = 35.65$

$NT = -5.33\sin 35.65 + 11.81\cos 35.65 = 6.5$ KN

Radial shear

$RS = V_x \cos\theta - H \sin\theta$

$= -5.33 \times \cos 35.65 - 11.81 \times \sin 35.65$

$= -11.2$ KN

(ii) BM, RS, NT (5m from RHS)

$x = 5$ m

$\theta = 28.65$

Bending moment

$Y_E = 3.75$

$M@E = 0$

Normal thrust

$NT = V_x Sin\theta + HCos\theta$

$V_x = 25$ KN

$NT = 25Sin28.65+50Cos28.65$

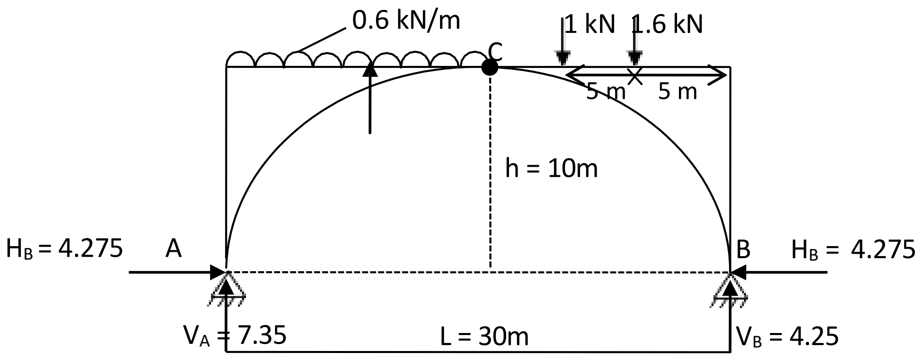
$= 55.87$ KN

Radial shear

RS $= V_x cos\theta - Hsin\theta = 25\times cos\ 28.65 - 50\times\ sin28.65$

$= -2.04$ KN

4. A 3-hinged arch has a span of 30m and a rise of 10m. The arch carries UDL of 0.6 kN/m on the left half of the span. It also carries 2 concentrated loads of 1.6 kN and 1 kN at 5 m and 10 m from the 'rt' end. Determine the reactions at the support. (sketch not given).



$$\sum F_x = 0$$

$$H_A - H_B = 0$$

$$H_A = H_B \qquad \text{- ---- (1)}$$

To find vertical reaction.

$$\sum F_y = 0$$

$$V_A + V_B = 0.6x15+1+1.6$$

$$= 11.6 \qquad \text{..........(2)}$$

$$\sum M_A = 0$$

$$-V_B\,x\,30 + 1.6\,x\,25 + 1x\,20 + (0.6\,x15)\,7.5 = 0$$

$$V_B = 4.25\ kN$$

$$V_A = 4.25 = 11.6$$

$$A_A = 7.35\ kN$$

To find horizontal reaction.

$M_C = 0$

$-1x5 - 1.6x10 + 4.25x15 - H_B x10 = 0$

$H_B = 4.275kN$

$H_A = 4.275kN$

OR

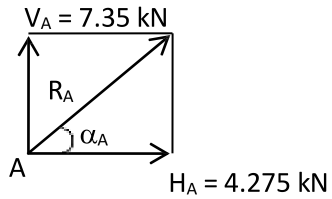
$M_C = 0$

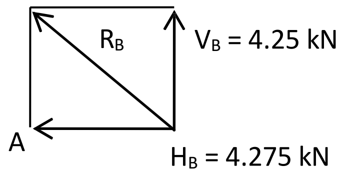
$7.375x15 - H_A x10 - (0.6x15)7.5$

$H_A = 4.275kN$

$H_B = 4.275kN$

To find total reaction





$R_A = \sqrt{H_A^{\ 2} = V_A^{\ 2}}$

$\sqrt{4.275^2 + 7.35^2}$

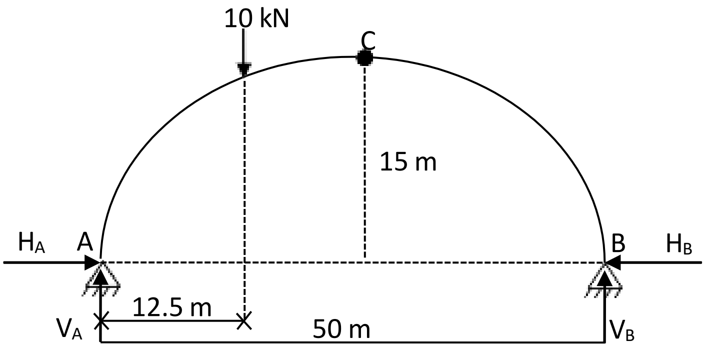
$= 8.5kN$

$\alpha_A = \tan^{-1}\left(\frac{V_A}{H_A}\right) = 59^0.82$

$R_B = \sqrt{H_B^{\ 2} + V_B^{\ 2} = 6.02kN}$

$\alpha_B = \tan^{-1}\left(\frac{V_B}{H_B}\right) = 44.83$

5. ***A 3-hinged parabolic arch of span 50m and rise 15m carries a load of 10kN at quarter span as shown in figure. Calculate total reaction at the hinges.***



$$\sum F_x = 0$$

$$H_A = H_B$$

To find vertical reaction:

$$\sum Fy = 0$$

$$V_A + V_B = 10 \text{ -- } (1)$$

$$\sum M_A = 0$$

$$-V_B \text{ x } 50 + 10 \text{ x} 12.5 = 0$$

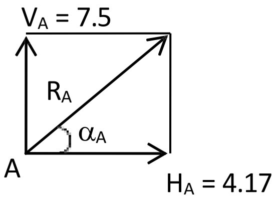
$$V_B = 2.5 \text{kN} \qquad V_A = 7.5 \text{kN}$$

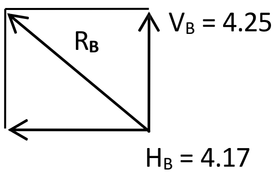
To find Horizontal reaction

$$M_C = 0$$

$$V_B \times 25 - H_B \times 15 = 0$$

To find total reaction.





$H_B = 4.17\ kN = H_A$

$R_A = \sqrt{4.17^2 + 7.5^2}$

$R_A = 8.581kN$

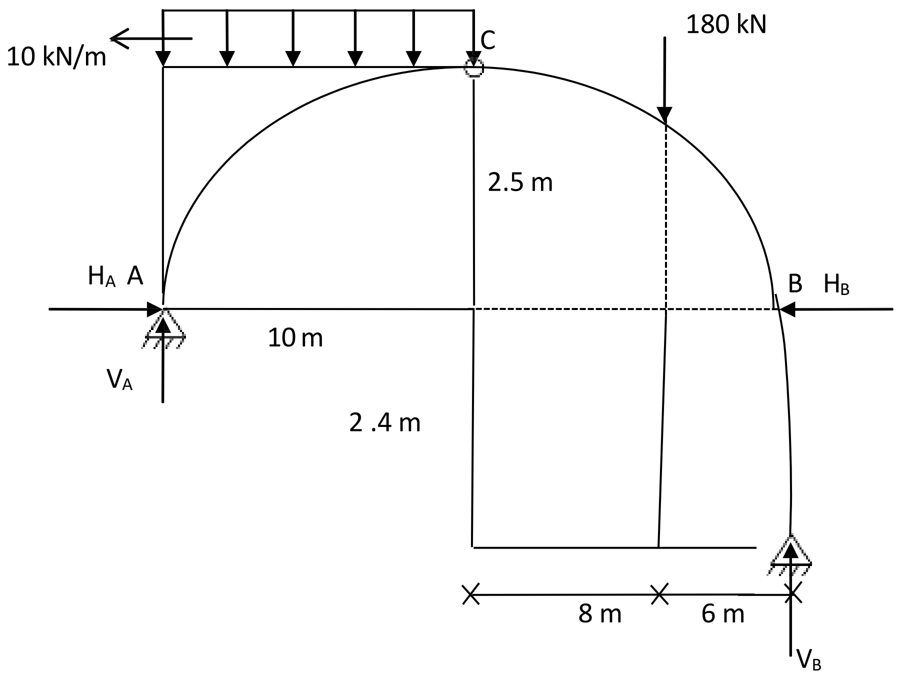
$\alpha_A = \tan^{-1}\left(\frac{V_A}{H_A}\right) = 60^0.92$

$R_B = \sqrt{H_A^{\ 2} + V_B^{\ 2}}$

$R_B = 4.861kN$

$\alpha_B = \tan^{-1}\left(\frac{V_B}{H_B}\right) = 30^0.94$

Problem: Determine the reaction components at supports A and B for 3-hinged arch shown in fig.



To find Horizontal reaction

$$\sum F_x = 0$$

$$H_A - H_B = 0$$

$$H_A = H_B \qquad \text{- ---- (1)}$$

To find vertical reaction.

$$\sum F_y = 0$$

$$V_A + V_B = 180 + 10 \text{ x} 10$$

$$V_A + V_B = 280 \qquad \text{..........(2)}$$

$$\sum M_A = 0$$

$$-V_B \text{x} 24 + H_B \text{ x } 2.4 + 180 \text{ x} 18 + 10 \text{ x } 10 \text{ x } 5 = 0$$

$$2.4 H_B - 24 V_B = -3740 \qquad \text{-- (3)}$$

$$H_B - 10 V_B = -1558.33$$

$$M_C = 0$$

$$-180x8 - V_B x14 - H_B x4.9 = 0$$

$$H_B x4.9 - V_B 14 = -1440 \text{--} (4)$$

$$-H_B + 2.857V_B = +293.87$$

Adding 2 and 3

$$-10V_B + 2.857V_B = -1558.33 + 293.87$$

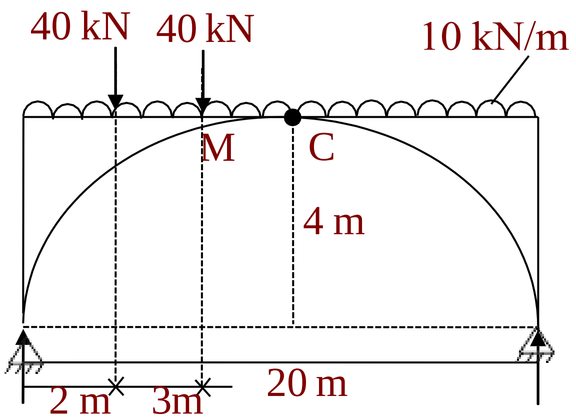
$$V_B = 177kN$$

$$V_A = 103kN$$

$$H_B - 10\ x177 = -1558.33$$

$$H_B = 211.67kN = H_A$$

A symmetrical 3-hinged parabolic arch has a span of 20m. It carries UDL of intensity 10 kNm over the entire span and 2 point loads of 40 kN each at 2m and 5m from left support. Compute the reactions. Also find BM, radial shear and normal thrust at a section 4m from left end take central rise as 4m.



$$\sum F_x = 0$$

$$H_A - H_B = 0$$

..........(1)

$$H_A = H_B$$

$$\sum F_y = 0$$

$$V_A + V_B - 40 - 40 - 10\,x\,20 = 0$$

..........(2)

$$V_A + V_B = 280$$

$$\sum M_A = 0$$

$$+40 \text{ x } 2 + 40 \text{ x } 5 + (10 \text{ x } 20)10 - V_B \text{ x } 20 = 0$$

$$V_B = 114 \text{ kN}$$

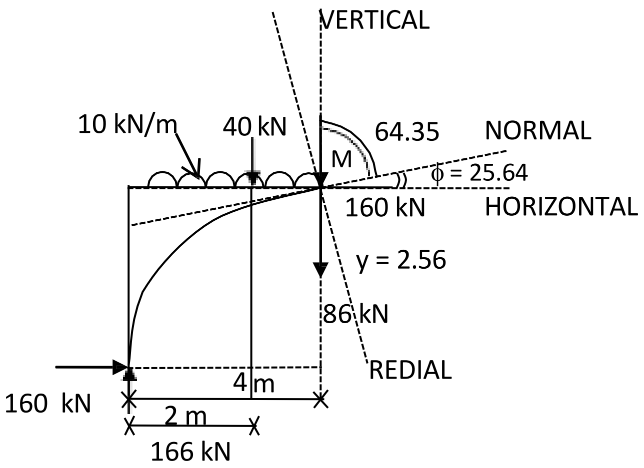
$$V_A = 166 \text{ kN}$$

$$M_c = 0$$

$$-(10 \text{ x} 10) 5 - H_B \text{ x } 4 + 114 \text{ x} 10 = 0$$

$$H_B = 160 \text{ kN}$$

$$H_A = 160 \text{ kN}$$



BM at M

= - 160 x 2.56

+ 166 x 4 – 40 x 2

- (10 x 4)2

= + 94.4 kNm

$$y = \frac{4hx}{L^2}(L - x)$$

$$= \frac{4 \text{ x } 4 \text{ x } 4}{20^2}(20 - 4)$$

$$y = 2.56\text{m}$$

$$\tan\phi = \frac{4h}{L^2}(L - 2x)$$

$$= \frac{4 \times 4}{20^2}(20 - 2 \times 4)$$

$$\phi = 25^0.64$$

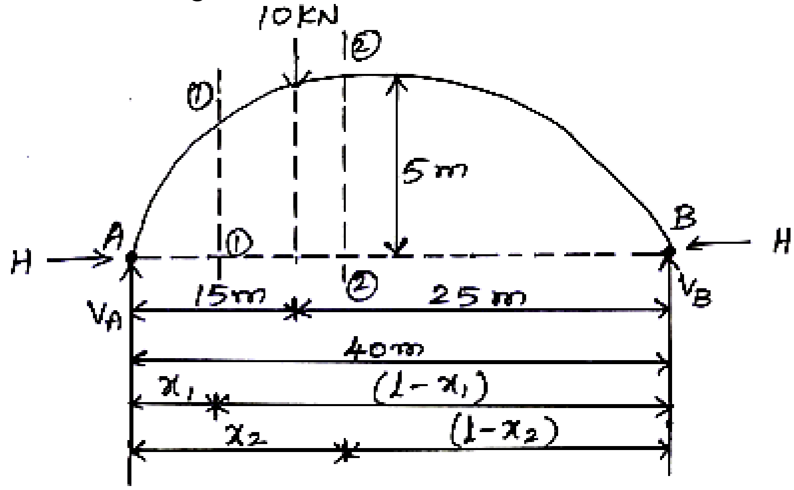
Normal thrust = N = + 160 Cos 25.64 + 86 Cos 64.36

= 181.46 kN

S = 160 Sin 25.64 - 86 x Sin 64.36

S = - 8.29 kN

1) A parabolic two hinged arch has a span of 40 m and a rise of 5 m. a concentrated load 10 kN acts at 15 m from the left support. The second moment of area varies as the secant of the inclination of the arch axis. Calculate the horizontal thrust and reactions at the hinge. Also calculate maximum bending moment at the section.



Step 1: Vertical Reactions :-

$\Sigma V = 0$

$$V_A + V_B = 10$$

Taking moment about A,

$$-V_B \times 40 + (10 \times 15) = 0$$

$$\boxed{V_B = 3.75 \text{ kN}}$$

$\therefore V_A = 10 - 3.75$

$$\boxed{V_A = 6.25 \text{ kN}}$$

Step 2: Horizontal Thrust (H):-

$$H = \frac{\int_0^l \mu y \, dx}{\int_0^l y^2 \, dx}$$

$$H = \frac{\int_0^{15} \mu_1 y \, dx + \int_{15}^{40} \mu_2 y \, dx}{\int_0^{40} y^2 \, dx} = \frac{Nr(1) + Nr(2)}{Dr}$$

where, M_1 = Beam bending moment in AX.

M_2 = Beam bending moment in XB.

i) Denominator :-

$$y = \frac{4y_c}{l^2} x(l-x) = \frac{4\times5}{(40)^2}(40x - x^2)$$

$$y = 0.5x - 0.0125x^2$$

$$\therefore Dr = \int_0^{40} y^2 dx$$

$$= \int_0^{40} (0.5x - 0.0125x^2)^2 dx$$

$$= \int_0^{40} (0.25x^2 + 1.56\times10^{-4} x^4 - 0.0125x^3) dx$$

$$= \left[\frac{0.25x^3}{3} + \frac{1.56\times10^{-4} x^5}{5} - \frac{0.0125x^4}{4} \right]_0^{40}$$

$$= \left[(5333.33 + 3194.88 - 8000) - 0 \right]$$

$$\boxed{Dr = 528.21}$$

ii) Numerator (1) :-

$$Nr(1) = \int_0^{15} M_1 y\, dx$$

here, $M_1 = V_A x_1 = 6.25x_1 = 6.25x$

$$Nr(1) = \int_0^{15} 6.25x(0.5x - 0.0125x^2) dx$$

$$= \int_0^{15} (3.125x^2 - 0.078x^3) dx$$

$$= \left[\frac{3.125x^3}{3} - \frac{0.078x^4}{4} \right]_0^{15}$$

$$= \left[(3515.63 - 987.18) - 0 \right]$$

$$\boxed{Nr(1) = 2528.45}$$

iii) Numerator (2) :-

$$Nr(2) = \int_{15}^{40} M_2 \, y \, dx$$

here, $M_2 = V_A \times x_2 - 10(x_2 - 15)$

$= 6.25x - 10x + 150$

$M_2 = 150 - 3.75x$

$$Nr(2) = \int_{15}^{40} (150 - 3.75x)(0.5x - 0.0125x^2)\, dx$$

$$= \int_{15}^{40} (75x - 1.875x^2 - 1.875x^2 + 0.047x^3)\, dx$$

$$= \int_{15}^{40} (75x - 3.75x^2 + 0.047x^3)\, dx$$

$$= \left[\frac{75x^2}{2} - \frac{3.75x^3}{3} + \frac{0.047x^4}{4} \right]_{15}^{40}$$

$$= \left[(60000 - 80000 + 30080) - (8437.5 - 4218.75 + 594.84) \right]$$

$$\boxed{Nr(2) = 5266.41}$$

$$\therefore H = \frac{Nr(1) + Nr(2)}{Dr} = \left(\frac{2528.45 + 5266.41}{528.21} \right)$$

$$\boxed{H = 14.76 \text{ KN}}$$

Step 3 : Reactions at A and B :-

$$R_A = \sqrt{V_A^2 + H^2} = \sqrt{(6.25)^2 + (14.76)^2}$$

$$\boxed{R_A = 16.03 \text{ KN}}$$

$$R_B = \sqrt{V_B^2 + H^2} = \sqrt{(3.75)^2 + (14.76)^2}$$

$$\boxed{R_B = 15.22 \text{ KN}}$$

Step 4 : Maximum Bending Moment:-

$$M_x = V_A \times 15 - H \times y$$

here,

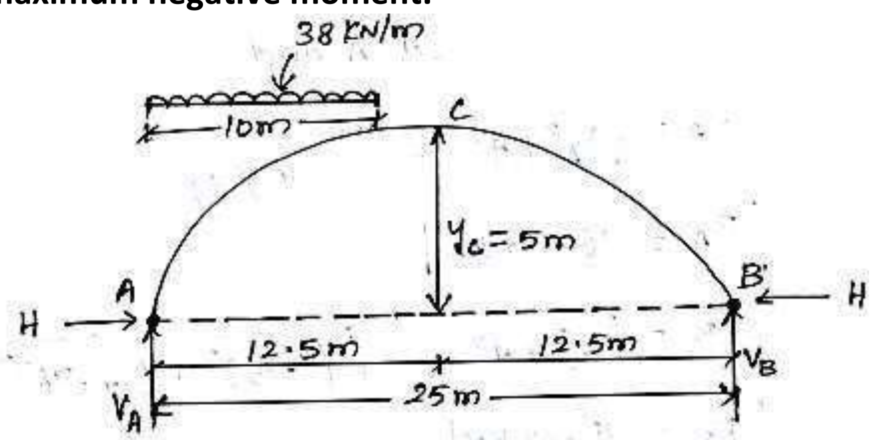
$$y = \frac{4y_c}{l^2} x(l-x) = \frac{4 \times 5}{(40)^2} \times 15(40-15)$$

$$\boxed{y = 4.68\ m}$$

$$\therefore \quad M_x = (6.25 \times 15) - (14.76 \times 4.68)$$

$$\boxed{M_{max} = 24.67\ KNm}$$

2. A two hinged parabolic arch of span 25 m and rise 5 m carries an udl of 38 kN/m covering a distance of 10 m from left end. Find the horizontal thrust, the reactions at the hinges and the maximum negative moment.



Step 1 : Vertical Reactions:-

$$\Sigma V = 0$$

$$V_A + V_B = 38 \times 10 = 380$$

Taking moment about A,

$$-V_B \times 25 + 38 \times \frac{(10)^2}{2} = 0$$

$$\boxed{V_B = 76\ KN}$$

$$\therefore \quad V_A = 380 - 76$$

$$\boxed{V_A = 304\ KN}$$

Step 2 : Horizontal thrust (H):-

$$H = \frac{\int_0^l (\mu_1 y + \mu_2 y)dx}{\int_0^l y^2 dx}$$

$$y = \frac{4y_c}{l^2}(lx - x^2) = \frac{4\times 5}{(25)^2}(25x - x^2)$$

$$y = 0.8x - 0.032x^2$$

i) Denominator :-

$$Dr = \int_0^{25} y^2 dx = \int_0^{25} (0.8x - 0.032x^2) dx$$

$$= \int_0^{25} (0.64x^2 + 1.02\times 10^{-3}x^4 - 0.051x^3) dx$$

$$= \left[\frac{0.64x^3}{3} + \frac{1.02\times 10^{-3}x^5}{5} - \frac{0.051x^4}{4}\right]_0^{25}$$

$$= [(3333.33 + 1992.18 - 4980.46) - 0]$$

$$\boxed{Dr = 345.05}$$

ii) Numerator (1) :- (loaded portion)

$$\mu_1 = V_A \times x - \frac{38\times x^2}{2} = 304x - 19x^2$$

$$Nr(1) = \int_0^{10} \mu_1 y\, dx = \int_0^{10} (304x - 19x^2)\times(0.8x - 0.032x^2)\, dx$$

$$= \int_0^{10} (243.2x^2 - 9.728x^3 - 15.2x^3 + 0.608x^4)\, dx$$

$$= \int_0^{10} (243.2x^2 - 24.93x^3 + 0.608x^4)\, dx$$

$$= \left[\frac{243.2x^3}{3} - \frac{24.93x^4}{4} + \frac{0.608x^5}{5}\right]_0^{10}$$

$$= [(81066.67 - 62325 + 12160) - 0]$$

$$\boxed{Nr(1) = 30901.67}$$

iii) Numerator (2) :-

$$M_2 = V_B \times (l-x) = 76(25-x) = 1900 - 76x$$

$$N_r(2) = \int_{10}^{25} M_2 \, y \, dx$$

$$= \int_{10}^{25} (1900 - 76x)(0.8x - 0.032x) \, dx$$

$$= \int_{10}^{25} (1520x - 60.8x^2 - 60.8x^2 + 2.43x^3) \, dx$$

$$= \left[\frac{1520x^2}{2} - \frac{121.6x^3}{3} + \frac{2.43x^4}{4} \right]_{10}^{25}$$

$$= \left[(475000 - 633333.33 + 237304.68) - (76000 - 40533.33 + 6075) \right]$$

$$= 78971.35 - 41541.67$$

$$\boxed{N_r(2) = 37429.68}$$

$$H = \frac{N_r(1) + N_r(2)}{D_r}$$

$$= \left(\frac{30901.67 + 37429.68}{345.05} \right)$$

$$\boxed{H = 198.03 \text{ kN}}$$

Step 3 : Resultant Reactions :-

$$R_A = \sqrt{V_A^2 + H^2} = \sqrt{(304)^2 + (198.03)^2}$$

$$\boxed{R_A = 362.81 \text{ kN}}$$

$$R_B = \sqrt{V_B^2 + H^2} = \sqrt{(76)^2 + (198.03)^2}$$

$$\boxed{R_B = 212.11 \text{ kN}}$$

here, $y = \frac{4y_c}{l^2} x(l-x)$

$= \frac{4 \times 5}{(25)^2} \times 18.75 \times 6.25$

$y = 3.75 m.$

$\therefore M_x = (76 \times 6.25) - (198.03 \times 3.75)$

$M_x = -267.61 \, kNm.$

Step 4 : Maximum Bending Moment:-

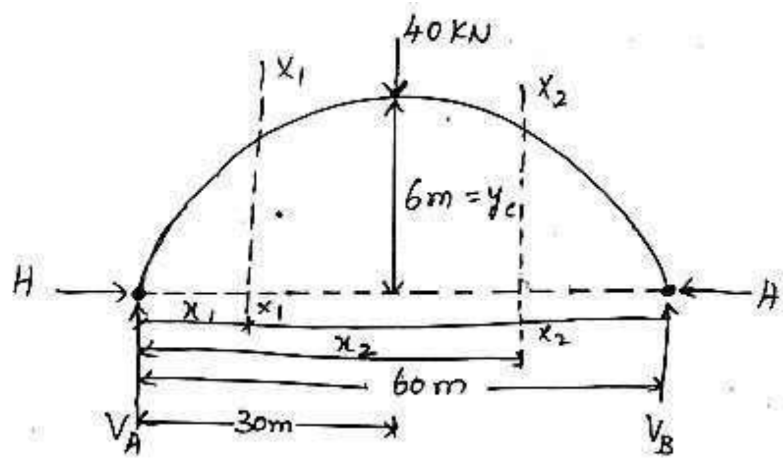
The maximum negative bending moment occurs at right of the span.

$x = \frac{3}{4} l = \frac{3}{4} \times 25 = 18.75 m$ from A.

$l - x = 25 - 18.75 = 6.25 m$ from B.

$B.M = V_B \times 6.25 - H \times y_x$

3. A parabolic two hinged arch of span 60 m and central rise of 6 m is subjected to a crown load of 40 kN. Allowing rib shortening and temperature rise of 20^o C, determine horizontal thrust, H. I_C = 6x10^5 cm^4, A_C = 1000 cm^2, E = 1x10^4 MPa, α = 11x10^{-6}/^{o} C, I = I_C sec θ.



Horizontal thrust, $H = H_1 + H_2$

here, H_1 = horizontal thrust under the load.

H_2 = horizontal thrust due to temperature rise

$$H_2 = \frac{l\alpha T EI}{\int_0^l y^2 ds}$$

$$H_1 = \frac{\int \mu y \, dx}{\int y^2 dx}$$

Step 1: Vertical Reactions:-

$$\Sigma V = 0$$

$$V_A + V_B = 40$$

Taking moment about A,

$$-V_B \times 60 + (40 \times 30) = 0$$

$$\boxed{V_B = 20 \text{ kN}}$$

$$V_A = 40 - 20$$

$$\boxed{V_A = 20 \text{ kN}}$$

Step 2: Horizontal thrust (H_1):-

$$H_1 = \frac{\int \mu y \, dx}{\int y^2 dx} = \frac{\int_0^{30} \mu_1 y dx + \int_{30}^{60} \mu_2 y dx}{\int_0^{60} y^2 dx}$$

$$H_1 = \frac{N_r(1) + N_r(2)}{D_r}$$

i) Denominator:-

$$D_r = \int_0^{60} y^2 dx$$

here, $y = \frac{4y_c}{l^2} x(l-x) = \frac{4 \times 6}{(60)^2} (lx - x^2)$

$$= \frac{24}{(60)^2} (60x - x^2)$$

$$\boxed{y = 0.4x - 0.0067x^2}$$

$$Dr = \int_0^{60} (0.4x - 0.0067x^2)^2 dx$$

$$= \int_0^{60} (0.16x^2 + 4.48\times10^{-5} x^4 - 0.0054x^3) dx$$

$$= \left[\frac{0.16x^3}{3} + \frac{4.48\times10^{-5} x^5}{5} - \frac{0.0054x^4}{4} \right]_0^6$$

$$= [(11520 + 6967.29 - 17496) - 0]$$

$$\boxed{Dr = 991.29}$$

ii) Numerator (1) :-

$$Nr(1) = \int_0^{30} M_1 y\, dx = \int_0^{30} (20x)(0.4x - 0.0067x^2) dx$$

$$= \int_0^{30} (8x^2 - 0.134x^3) dx$$

$$= \left(\frac{8x^3}{3} - \frac{0.134x^4}{4} \right)_0^{30}$$

$$= [(72000 - 27135) - 0]$$

$$\boxed{Nr(1) = 44865}$$

iii) Numerator(2) :-

$$Nr(2) = \int_{30}^{60} M_2 y\, dx$$

here, $M_2 = 20x - 40x + 1200$

$M_2 = 1200 - 20x$

$$Nr(2) = \int_{30}^{60} (1200 - 20x)(0.4x - 0.0067x^2) dx$$

$$= \int_{30}^{60} (480x - 8.04x^2 - 8x^2 + 0.134x^3) dx$$

$$= \int_{30}^{60} (480x - 16.04x^2 + 0.134x^3) dx$$

$$= \left[\frac{480x^2}{2} - \frac{16.04x^3}{3} + \frac{0.134x^4}{4} \right]_{30}^{60}$$

$$= \left[(864000 - 1154880 + 434160) - (216000 - 144360 + 27135) \right]$$

$$\boxed{N_r(2) = 44505}$$

$$\therefore H_1 = \frac{N_r(1) + N_r(2)}{D_r}$$

$$= \left(\frac{44865 + 44505}{991.29} \right)$$

$$\therefore \boxed{H_1 = 90.16 \text{ kN}}$$

Step 3: Increased Horizontal thrust :-

$$H_2 = \frac{l \alpha T E I}{\int_0^l y^2 ds}$$

here,

$l = 60\ m$;

$\alpha = 11 \times 10^{-6} /°c$.

$T = 20°c$.

$E = 1 \times 10^4 MPa = 1 \times 10^{10} N/m^2$.

$I = I_c \sec\theta$

$I_c = 6 \times 10^5\ cm^4 = 6 \times 10^5 \times 10^{-8} = 0.006\ m^4$.

$$y = \frac{4r}{l^2} (lx - x^2)$$

$$\theta = \frac{dy}{dx} = \frac{4r}{l^2} (l - 2x)$$

$$= \frac{4\times 6}{(60)^2}(60-2(30))$$

$$\theta = 0.$$

$$\therefore \quad I = I_c \sec\theta$$

$$= \frac{0.006}{\cos(0)}$$

$$I = 0.006\ m^4$$

$$y = \frac{4y_c}{l^2}(lx - x^2)$$

$$= \frac{4\times 6}{(60)^2}(60x - x^2)$$

$$y = 0.4x - 0.0067x^2$$

$$H_2 = \frac{l\alpha T E I}{\int_0^{60} y^2\,dx}$$

here,

$$Dr = \int_0^{60} y^2\,dx = \int_0^{60} (0.4x - 0.0067x^2)^2\,dx$$

$$= \int_0^{60} (0.16x^2 + 4.489\times10^{-5}x^4 - 5.36\times10^{-3}x^3)\,dx$$

$$= \left[\frac{0.16x^3}{3} + \frac{4.489\times10^{-5}x^5}{5} - \frac{5.36\times10^{-3}x^4}{4}\right]_0^{60}$$

$$= \left[(11520 + 6981.29 - 17366.4) - 0\right]$$

$$\boxed{Dr = 1134.89}$$

$$\therefore H_2 = \left(\frac{60\times 11\times10^{-6}\times 20\times 1\times10^{10}\times 0.006}{1134.89}\right)$$

$$H_2 = 697.86\ N$$

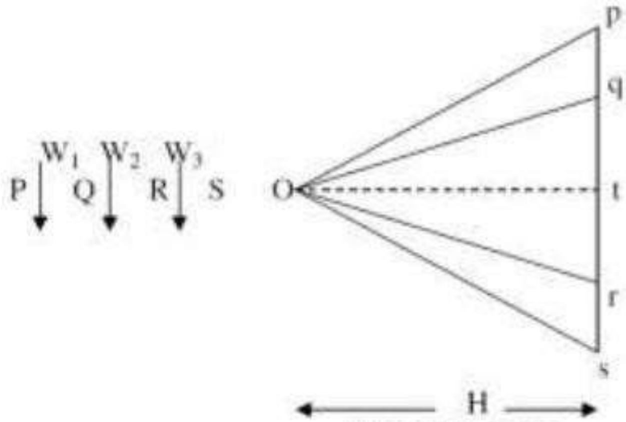
$$\boxed{H_2 = 0.697\ KN}$$

$$H = H_1 + H_2 = 90.16 + 0.697$$

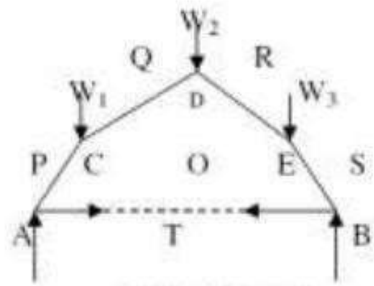
$$\boxed{H = 90.857\ KN}$$

What is a linear arch?

If an arch is to take loads, say W_1, W_2, and W_3 (fig) and a Vector diagram and funicular polygon are plotted as shown, the funicular polygon is known as the linear arch or theoretical arch.



Vector Diagram



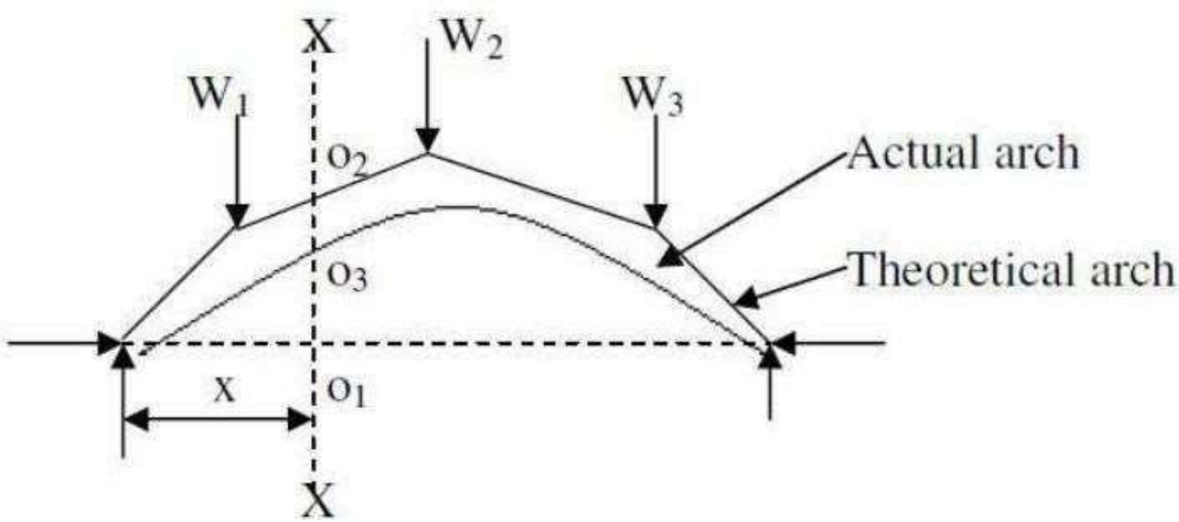
Space Diagram

✦ The polar distance „ot‟ represents the horizontal thrust.
✦ The links AC, CD, DE, and EB will be under compression and there will be no bending moment.
✦ If an arch of this shape ACDEB is provided, there will be no bending moment.
✦ For a given set of vertical loads W1, W2…..etc., we can have any number of linear arches depending on where we choose „O‟ or how much horizontal thrust (or) we choose to introduce.

State Eddy's theorem.

✦ Eddy‟s theorem states that "The bending moment at any section of an arch is proportional to the vertical intercept between the linear arch (or theoretical arch) and the centre line of the actual arch."

✦ BM_x = Ordinate O_2O_3 x scale factor



Distinguish between two hinged and three hinged arches

Sl. No	*Two hinged arches*	*Three hinged arches*
1	Statically indeterminate to first degree	Statically determinate
2	Might develop temperature stresses	Increase in temperature causes increase in Central rise. No stresses.
3	Structurally more efficient	Easy to analyse. But in costruction, the central hinge may involve additional expenditure.
4	Will develop stresses due to sinking of supports	Since this is determinate, no stresses due to support sinking.

Rib-shortening in the case of arches.

✦ In a two hinged arch, the normal thrust which is a compressive force along the axis of the arch will shorten the rib of the arch.

✦ This in turn will release part of the horizontal thrust.

✦ Normally, this effect is not considered in the analysis (in the case of two hinged arches).

✦ Depending upon the importance of the work we can either take into account or omit the effect of rib shortening.

✦ This will be done by considering (or omitting) strain energy due to axial compression along with the strain energy due to bending in evaluating H.

Strain energy due to bending (U_b)

$$U_b = \int_0^s \frac{M^2}{2EI} ds$$

Where,

M	=	*Bending moment*
E	=	*Young's modulus of the arch material*
I	=	*Moment of inertia of the arch cross section*
s	=	*Length of the centreline of the arch*

Strain energy due to axial compression (U_a)

$$U_a = \int_0^s \frac{N^2}{2AE} ds$$

Where,

M	=	*Bending moment*
N	=	*Axial compression.*
A	=	*Cross sectional area of the arch*
E	=	*Young's modulus of the arch material*
s	=	*Length of the centreline of the arch*

Total strain energy of the arch

$$U = \int_0^s \frac{M^2}{2EI} ds + \int_0^s \frac{N^2}{2AE} ds$$